Chapter 1: Basic Principles of Enumeration

Theorem 1.2. *For every $n \geq 1$, the number of lists of an n-element set S is*

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

Theorem 1.3. *For every $n \geq 0$, the number of subsets of an n-element set is 2^n.*

Theorem 1.4. *For $n, k \geq 0$, the number of partial lists of length k of an n-element set is $n(n-1)\cdots(n-k+2)(n-k+1)$.*

Theorem 1.5. *For $0 \leq k \leq n$, the number of k-element subsets of an n-element set is*

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 1.9. *For any $n \geq 0$ and $t \geq 1$, the number of n-element multisets with elements of t types is*

$$\binom{n+t-1}{t-1}.$$

Theorem 1.15 (Inclusion/Exclusion). *Let $A_1, A_2, \ldots, A_m$ be finite sets. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{\varnothing \neq S \subseteq \{1,\ldots,m\}} (-1)^{|S|-1} |A_S|.$$

Proposition 1.11. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ be functions between two sets $\mathcal{A}$ and $\mathcal{B}$. Assume the following.*

- *For all $a \in \mathcal{A}$, $g(f(a)) = a$.*
- *For all $b \in \mathcal{B}$, $f(g(b)) = b$.*

Then both f and g are bijections. Moreover, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have $f(a) = b$ if and only if $g(b) = a$.

Chapter 2: The Idea of Generating Series

Theorem 2.2 (The Binomial Theorem). *For any natural number $n \in \mathbb{N}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Theorem 2.4 (The Binomial Series). *For any positive integer $t \geq 1$,*

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n.$$

Proposition 2.7. *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$, and let*

$$\Phi_{\mathcal{A}}(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

For every $n \in \mathbb{N}$, the number of elements of $\mathcal{A}$ of weight n is $a_n = |\mathcal{A}_n|$.

Lemma 2.10 (The Sum Lemma.). *Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint sets, so that $\mathcal{A} \cap \mathcal{B} = \varnothing$. Assume that $\omega : (\mathcal{A} \cup \mathcal{B}) \to \mathbb{N}$ is a weight function on the union of $\mathcal{A}$ and $\mathcal{B}$. We may regard ω as a weight function on each of $\mathcal{A}$ or $\mathcal{B}$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{A}\cup\mathcal{B}}(x) = \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x).$$

Lemma 2.11 (The Infinite Sum Lemma.). *Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ be pairwise disjoint sets (so that $\mathcal{A}_i \cap \mathcal{A}_j = \varnothing$ if $i \neq j$), and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. Assume that $\omega : \mathcal{B} \to \mathbb{N}$ is a weight function. We may regard ω as a weight function on each of the sets $\mathcal{A}_j$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{B}}(x) = \sum_{j=0}^{\infty} \Phi_{\mathcal{A}_j}(x).$$

Lemma 2.12 (The Product Lemma.). *Let $\mathcal{A}$ and $\mathcal{B}$ be sets with weight functions $\omega : \mathcal{A} \to \mathbb{N}$ and $\nu : \mathcal{B} \to \mathbb{N}$, respectively. Define $\eta : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by putting $\eta(\alpha, \beta) = \omega(\alpha) + \nu(\beta)$ for all $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. Then η is a weight function on $\mathcal{A} \times \mathcal{B}$, and*

$$\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) = \Phi^{\omega}_{\mathcal{A}}(x) \cdot \Phi^{\nu}_{\mathcal{B}}(x).$$

Taylor Series (Maclaurin Series)

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$

Lemma 2.13. *Let $\mathcal{A}$ be a set with weight function $\omega : \mathcal{A} \to \mathbb{N}$, and define $\mathcal{A}^*$ and $\omega^* : \mathcal{A}^* \to \mathbb{N}$ as above. Then ω^* is a weight function on $\mathcal{A}^*$ if and only if there are no elements in $\mathcal{A}$ of weight zero (that is, $\mathcal{A}_0 = \varnothing$).*

Lemma 2.14 (The String Lemma.). *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$ such that there are no elements of $\mathcal{A}$ of weight zero. Then*

$$\Phi_{\mathcal{A}^*}(x) = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}.$$

Theorem 2.17. *Let $P = \{1, 2, 3, \ldots\}$ be the set of positive integers.*

(a) The set $\mathcal{C}$ of all compositions is $\mathcal{C} = P^*$.
(b) The generating series for $\mathcal{C}$ with respect to size is

$$\Phi_{\mathcal{C}}(x) = 1 + \frac{x}{1-2x}.$$

(c) For each $n \in \mathbb{N}$, the number of compositions of size n is

$$|\mathcal{C}_n| = \begin{cases} 1 & \text{if } n = 0, \\ 2^{n-1} & \text{if } n \geq 1. \end{cases}$$

Lemma 2.25. *For any nonempty set T,*

$$\sum_{\varnothing \neq S \subseteq T} (-1)^{|S|-1} = 1.$$

Theorem 2.26 (Inclusion/Exclusion). *Let $A_1, A_2, \ldots, A_m$ be finite sets. Then*

$$|A_1 \cup \cdots \cup A_m| = \sum_{\varnothing \neq S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|-1} |A_S|.$$

Proposition 2.23. *Let $\mathcal{U}$ be the set of pairs (n, A) in which $n \in \mathbb{N}$ is a natural number and $A \subseteq \{1, 2, \ldots, n\}$ is a subset. Let $\mathcal{C} \smallsetminus \{\varepsilon\}$ be the set of nonempty compositions. There is a bijection $\mathcal{U} \rightleftharpoons \mathcal{C} \smallsetminus \{\varepsilon\}$ between these two sets.*

Chapter 3: Binary Strings

Lemma 3.9 (Unambiguous Expression). *Let R and S be unambiguous expressions producing the sets $\mathcal{R}$ and $\mathcal{S}$, respectively.*

- *The expressions ε and 0 and 1 are unambiguous.*
- *The expression $\mathsf{R} \smile \mathsf{S}$ is unambiguous if and only if $\mathcal{R} \cap \mathcal{S} = \varnothing$, so that $\mathcal{R} \cup \mathcal{S}$ is a disjoint union of sets.*
- *The expression RS is unambiguous if and only if there is a bijection $\mathcal{RS} \rightleftharpoons \mathcal{R} \times \mathcal{S}$ between the concatenation product $\mathcal{RS}$ and the Cartesian product $\mathcal{R} \times \mathcal{S}$. In other words, for every string $\alpha \in \mathcal{RS}$ there is exactly one way to write $\alpha = \rho\sigma$ with $\rho \in \mathcal{R}$ and $\sigma \in \mathcal{S}$.*
- *The expression R^* is unambiguous if and only if each of the concatenation products R^k is unambiguous and the union $\bigcup_{k=0}^{\infty} \mathcal{R}^k$ is a disjoint union of sets.*

Theorem 3.13. *Let R be a regular expression producing the rational language $\mathcal{R}$ and leading to the rational function $R(x)$. If R is an unambiguous expression for $\mathcal{R}$ then $R(x) = \Phi_{\mathcal{R}}(x)$, the generating series for $\mathcal{R}$ with respect to length.*

Proposition 3.17 (Block Decompositions.). *The regular expressions*

$$0^*(1^*10^*0)^*1^* \quad \textit{and} \quad 1^*(0^*01^*1)^*0^*$$

are unambiguous expressions for the set $\{0,1\}^*$ of all binary strings. They produce each binary string block by block.

Theorem 3.26. *Let $\kappa \in \{0,1\}^*$ be a nonempty string of length n, and let $\mathcal{A} = \mathcal{A}_\kappa$ be the set of binary strings that avoid κ. Let $\mathcal{C}$ be the set of all nonempty suffixes γ of κ such that $\kappa\gamma = \eta\kappa$ for some nonempty prefix η of κ. Let $C(x) = \sum_{\gamma \in \mathcal{C}} x^{\ell(\gamma)}$. Then*

$$A(x) = \frac{1 + C(x)}{(1-2x)(1+C(x)) + x^n}.$$

Chapter 4: Recurrence Relations

Theorem 4.8. *Let $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ be a sequence of complex numbers, and let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. The following are equivalent.*

(a) *The sequence $\mathbf{g}$ satisfies a homogeneous linear recurrence relation*

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0 \text{ for all } n \geq N,$$

with initial conditions $g_0, g_1, \ldots, g_{N-1}$.

(b) *The series $G(x) = P(x)/Q(x)$ is a quotient of two polynomials. The denominator is*

$$Q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

and the numerator is $P(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$, in which

$$b_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$$

for all $0 \leq k \leq N-1$, with the convention that $g_n = 0$ for all $n < 0$.

Theorem 4.12 (Partial Fractions). *Let $G(x) = P(x)/Q(x)$ be a rational function in which $\deg P < \deg Q$ and the constant term of $Q(x)$ is 1. Factor the denominator to obtain its inverse roots:*

$$Q(x) = (1 - \lambda_1 x)^{d_1} (1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}$$

in which $\lambda_1, \ldots, \lambda_s$ are distinct nonzero complex numbers and $d_1 + \cdots + d_s = d = \deg Q$. Then there are d complex numbers:

$$C_1^{(1)}, C_1^{(2)}, \ldots, C_1^{(d_1)};\ C_2^{(1)}, C_2^{(2)}, \ldots, C_2^{(d_2)};\ \ldots;\ C_s^{(1)}, C_s^{(2)}, \ldots, C_s^{(d_s)}$$

(which are uniquely determined) such that

$$G(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1 - \lambda_i x)^j}.$$

Theorem 4.18. *Let $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ be a sequence of complex numbers. The following are equivalent.*

(a) *The sequence $\mathbf{g}$ satisfies a homogeneous linear recurrence relation (with initial conditions).*

(b) *The sequence $\mathbf{g}$ satisfies a possibly inhomogeneous linear recurrence relation (with initial conditions) in which the RHS is an eventually polyexp function.*

(c) *The generating series $G(x) = \sum_{n=0} g_n x^n$ is a rational function (a quotient of polynomials in x).*

(d) *The sequence $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ is an eventually polyexp function.*

Definition 4.20. For any complex number $\alpha \in \mathbb{C}$ and nonnegative integer $k \in \mathbb{N}$, the *k-th binomial coefficient of α* is

$$\binom{\alpha}{k} = \frac{1}{k!} (\alpha)(\alpha - 1) \cdots (\alpha - k + 1).$$

Theorem 4.21 (The Binomial Series). *For any complex number $\alpha \in \mathbb{C}$,*

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Proposition 4.22. $\sqrt{1 - 4x} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^k.$

Theorem 4.14. *Let $\mathbf{g} = (g_0, g_1, g_2)$ be a sequence of complex numbers, and let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. Assume that the equivalent conditions of Theorem 4.8 hold, and that*

$$G(x) = R(x) + \frac{P(x)}{Q(x)}$$

for some polynomials $P(x)$, $Q(x)$, and $R(x)$ with $\deg P(x) < \deg Q(x)$ and $Q(0) = 1$. Factor $Q(x)$ to obtain its inverse roots and their multiplicities:

$$Q(x) = (1 - \lambda_1 x)^{d_1} (1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}.$$

Then there are polynomials $p_i(n)$ for $1 \leq i \leq s$, with $\deg p_i(n) < d_i$, such that for all $n > \deg R(x)$,

$$g_n = p_1(n) \lambda_1^n + p_2(n) \lambda_2^n + \cdots + p_s(n) \lambda_s^n.$$

Part II: Chapter 4: Introduction to Graph Theory

Theorem 4.3.1. *For any graph G we have*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Corollary 4.3.2. *The number of vertices of odd degree in a graph is even.*

Corollary 4.3.3. *The average degree of a vertex in the graph G is*

$$\frac{2|E(G)|}{|V(G)|}.$$

Theorem 4.6.2. *If there is a walk from vertex x to vertex y in G, then there is a path from x to y in G.*

Corollary 4.6.3. *Let x, y, z be vertices of G. If there is a path from x to y in G and a path from y to z in G, then there is a path from x to z in G.*

Theorem 4.6.4. *If every vertex in G has degree at least 2, then G contains a cycle.*

Theorem 4.8.2. *Let G be a graph and let v be a vertex in G. If for each vertex w in G there is a path from v to w in G, then G is connected.*

Theorem 4.8.5. *A graph G is not connected if and only if there exists a proper nonempty subset X of $V(G)$ such that the cut induced by X is empty.*

Theorem 4.9.2. *Let G be a connected graph. Then G has an Eulerian circuit if and only if every vertex has even degree.*

Lemma 4.10.2. *If $e = \{x, y\}$ is a bridge of a connected graph G, then $G - e$ has precisely two components; furthermore, x and y are in different components.*

Theorem 4.10.3. *An edge e is a bridge of a graph G if and only if it is not contained in any cycle of G.*

Corollary 4.10.4. *If there are two distinct paths from vertex u to vertex v in G, then G contains a cycle.*

Chapter 5: Tree

Lemma 5.1.3. *If u and v are vertices in a tree T, then there is a unique u, v-path in T*

Lemma 5.1.4. *Every edge of a tree T is a bridge.*

Theorem 5.1.5. *If T is a tree, then $|E(T)| = |V(T)| - 1$.*

Corollary 5.1.6. *If G is a forest with k components, then $|E(G)| = |V(G)| - k$.*

Theorem 5.1.8. *A tree with at least two vertices has at least two leaves.*

Theorem 5.2.1. *A graph G is connected if and only if it has a spanning tree.*

Corollary 5.2.2. *If G is connected, with p vertices and $q = p - 1$ edges, then G is a tree.*

Corollary

Corollary: Let G be a graph with n vertices. If any 2 of the following 3 conditions hold, then G is a tree.

① G is connected. ② G has no cycles. ③ G has n-1 edges.

Note: ①+② is by definition. ①+③ is by corollary. ②+③ we leave as an exercise.

Theorem 5.2.3. *If T is a spanning tree of G and e is an edge not in T, then $T + e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T + e - e'$ is also a spanning tree of G.*

Theorem 5.2.4. *If T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also a spanning tree of G.*

Theorem 5.6.1. *Prim's algorithm produces a minimum spanning tree for G.*

Corollary 7.5.4

Theorem (Corollary 7.5.4 in course notes): Every planar graph has a vertex of degree at most 5.

Chapter 7: Planar Graphs

Theorem 7.1.2. *If we have a planar embedding of a connected graph G with faces $f_1, \ldots, f_s$, then*

$$\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|.$$

Faceshaking Lemma

Jordan Curve Theorem:
Every planar embedding of s cycle separates the plane into two parts, one on inside, one on the outside.

Theorem 7.2.1. *(Euler's Formula) Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then*

$$p - q + f = 2.$$

Theorem 7.3.1. *A graph is planar if and only if it can be drawn on the surface of a sphere.*

Theorem 7.4.1. *There are exactly five platonic graphs.*

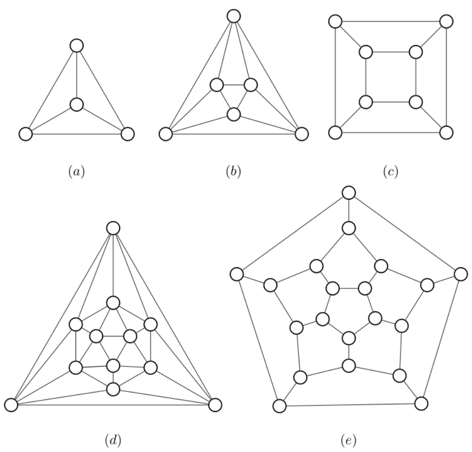

Figure 7.9: (a) the tetrahedron; (b) the octahedron; (c) the cube;(d) the icosahedron; (e) the dodecahedron

Lemma 7.5.1. *If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle.*

Show too many edges

Lemma 7.5.2. *Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d^*, then $(d^* - 2)q \le d^*(p-2)$.*

Theorem 7.5.3. *In a planar graph G with $p \ge 3$ vertices and q edges, we have*

$$q \le 3p - 6.$$

Corollary 7.5.4. *K_5 is a not planar.*

Corollary 7.5.5. *A planar graph has a vertex of degree at most five.*

Theorem 7.5.6. *In a bipartite planar graph G with $p \ge 3$ vertices and q edges, we have*

$$q \le 2p - 4.$$

Lemma 7.5.7. *$K_{3,3}$ is not planar.*

Kuratowski's Theorem

Theorem 7.6.1. *A graph is not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$.*

Theorem 7.7.2. *A graph is 2-colourable if and only if it is bipartite.*

Theorem 7.7.3. *K_n is n-colourable, and not k-colourable for any $k < n$.*

Theorem 7.7.4. *Every planar graph is 6-colourable.*

Theorem 7.7.6. *Every planar graph is 5-colourable.*

Theorem 7.7.7. *Every planar graph is 4-colourable.*

Lemma 7.4.3. *If G is a platonic graph with p vertices, q edges and f faces, where each vertex has degree d and each face degree d^*, then*

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

and $p = 2q/d$ and $f = 2q/d^*$.

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Lemma 7.4.2. *Let G be a planar embedding with p vertices, q edges and s faces, in which each vertex has degree $d \ge 3$ and each face has degree $d^* \ge 3$. Then (d, d^*) is one of the five pairs*

$$\{(3,3), (3,4), (4,3), (3,5)\,(5,3)\}.$$

Chapter 8: Matchings

Lemma 8.1.1. *If M has an augmenting path, it is not a maximum matching.*

Lemma 8.2.1. *If M is a matching of G and C is a cover of G, then $|M| \le |C|$.*

Lemma 8.2.2. *If M is matching and C is a cover and $|M| = |C|$, then M is a maximum matching and C is a minimum cover.*

Theorem 8.3.1. *(König's Theorem) In a bipartite graph the maximum size of a matching is the minimum size of a cover.*

Let X_0 be the set of vertices in A not saturated by M and let Z denote the set of vertices in G that are joined by to a vertex in X_0 by an alternating path. If $v \in Z$ we use $P(v)$ to denote an alternating path that joins v to X_0. Now define:

(a) $X = A \cap Z$.

(b) $Y = B \cap Z$.

Lemma 8.3.2. *Let M be a matching of bipartite graph G with bipartition A, B, and let X and Y be as defined above. Then:*

(a) There is no edge of G from X to $B \setminus Y$;

(b) $C = Y \cup (A \setminus X)$ is a cover of G;

(c) There is no edge of M from Y to $A \setminus X$;

(d) $|M| = |C| - |U|$ where U is the set of unsaturated vertices in Y;

(e) There is an augmenting path to each vertex in U.

Bipartite matching algorithm

Step 0. Let M be any matching of G.
Step 1. Set $\hat{X} = \{v \in A : v \text{ is unsaturated}\}$, set $\hat{Y} = \emptyset$, and set $\mathrm{pr}(v)$ to be undefined for all $v \in V(G)$.
Step 2. For each vertex $v \in B \setminus \hat{Y}$ such that there is an edge $\{u, v\}$ with $u \in \hat{X}$, add v to $\hat{Y}$ and set $\mathrm{pr}(v) = u$.
Step 3. If Step 2 added no vertex to $\hat{Y}$, return the maximum matching M and the minimum cover $C = \hat{Y} \cup (A \setminus \hat{X})$, and stop.

Step 4. If Step 2 added an unsaturated vertex v to $\hat{Y}$, use pr values to trace an augmenting path from v to an unsaturated element of $\hat{X}$, use the path to produce a larger matching M', replace M by M', and go to Step 1.
Step 5. For each vertex $v \in A \setminus \hat{X}$ such that there is an edge $\{u, v\} \in M$ with $u \in \hat{Y}$, add v to $\hat{X}$ and set $\mathrm{pr}(v) = u$. Go to Step 2.

Theorem 8.4.1. *(Hall's Theorem) A bipartite graph G with bipartition A, B has a matching saturating every vertex in A, if and only if every subset D of A satisfies*

$$|N(D)| \ge |D|.$$

Corollary 8.6.1. *A bipartite graph G with bipartition A, B has a perfect matching if and only if $|A| = |B|$ and every subset D of A satisfies*

$$|N(D)| \ge |D|.$$

Theorem 8.6.2. *If G is a k-regular bipartite graph with $k \ge 1$, then G has a perfect matching.*

Theorem 8.7.1. *A bipartite graph with maximum degree Δ has an edge Δ-colouring.*

Lemma 8.7.2. *Let G be a bipartite graph having at least one edge. Then G has a matching saturating each vertex of maximum degree.*

Theorem 8.8.1. *Let G be a graph with q edges, and suppose k, m are positive integers such that*

(a) G has an edge k-colouring;

(b) $q \le km$.

Then G has an edge k-colouring in which every colour is used at most m times.

Corollary 8.8.2. *In a bipartite graph G, there is an edge k-colouring in which each colour is used at most m times if and only if*

(a) $\Delta \le k$, and

(b) $q \le km$.

Corollary: The edges of a k-regular bipartite graph can be partitioned into k perfect matchings.

Corollary: A bipartite graph G with m edges and maximum degree d has a matching of size at least $\frac{m}{d}$.