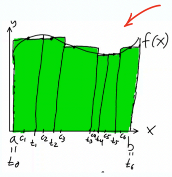
1.2 - Riemann Sums and the Definite Integral



what it does

DEFINITION **Riemann Sum**

Given a bounded function f on $[a, b]$, a partition P

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$, and a set $\{c_1, c_2, \ldots, c_n\}$ (points) where $c_i \in [t_{i-1}, t_i]$, then a *Riemann sum* for f with respect to P is a sum of the form

each interval has its corresponding points.

Area under the curve that are estimated with these green rectangle.

$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i.$$

DEFINITION **Regular n-Partition**

Given an interval $[a, b]$ and an $n \in \mathbb{N}$, the *regular n−partition* of $[a, b]$ is the partition $P^{(n)}$ with

$$a = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$ where each subinterval has the *same* length $\Delta t_i = \frac{b-a}{n}$.

DEFINITION **Left-hand Riemann Sum**

The *left-hand Riemann sum* for f with respect to the partition P is the Riemann sum L obtained from P by choosing c_i to be t_{i-1}, the left-hand endpoint of $[t_{i-1}, t_i]$. That is

$$L = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i.$$

If $P^{(n)}$ is the regular n-partition, we denote the left-hand Riemann sum by

$$\begin{aligned} L_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i &= \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n} \\ &= \sum_{i=1}^{n} f\left(a + (i-1)\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \end{aligned}$$

DEFINITION **Right-hand Riemann Sum**

The *right-hand Riemann sum* for f with respect to the partition P is the Riemann sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is

$$R = \sum_{i=1}^{n} f(t_i)\Delta t_i.$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by

$$\begin{aligned} R_n = \sum_{i=1}^{n} f(t_i)\Delta t_i &= \sum_{i=1}^{n} f(t_i)\boxed{\frac{b-a}{n}} \\ &= \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \end{aligned}$$

DEFINITION **Definite Integral**

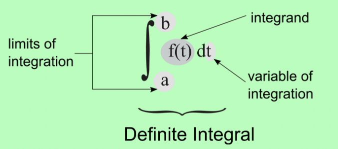
We say that a bounded function f is *integrable* on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\boxed{\lim_{n\to\infty} \|P_n\| = 0}$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have

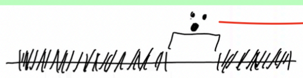
$$\lim_{n\to\infty} S_n = I.$$

In this case, we call I the integral of f over $[a, b]$ and denote it by[3]

$$\boxed{\int_a^b f(t)\,dt}$$

The points a and b are called the *limits of integration* and the function $f(t)$ is called the *integrand*. The variable t is called the *variable of integration*.





Because if this gap (p) is not going to 0, we will never be able to estimate the area.

NOTE

The variable of integration is sometimes called a *dummy variable* in the sense that if we were to replace t's by x's everywhere, we would not change the value of the integral.

That is: $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(☺)\,d☺$

REM 1 **Integrability Theorem for Continuous Functions**

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$ ← Right hand one

and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

Let's examine a "nice" choice: one where the partition is regular and where we just pick the C_i's to be the right-hand endpoints!

REMARK

This theorem also holds if f is bounded and has finitely many discontinuities on $[a, b]$. The proof of this theorem is beyond the scope of this course.

1.3 - Properties of the Definite Integral

THEOREM 2 **Properties of Integrals**

Assume that f and g are integrable on the interval $[a, b]$. Then:

i) For any $c \in \mathbb{R}$, $\int_a^b c\, f(t)\,dt = c\int_a^b f(t)\,dt$. Factor out real numbers.

ii) $\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$. Integral of sum = Sum of integral

iii) If $m \le f(t) \le M$ for all $t \in [a, b]$, then $m(b-a) \le \int_a^b f(t)\,dt \le M(b-a)$.

iv) If $0 \le f(t)$ for all $t \in [a, b]$, then $0 \le \int_a^b f(t)\,dt$. special case of (iii)

v) If $g(t) \le f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)\,dt \le \int_a^b f(t)\,dt$.

vi) The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|\,dt$.

DEFINITION $\int_a^a f(t)\,dt$ **[Identical Limits of Integration]** More properities

Let $f(t)$ be defined at $t = a$. Then we define

$$\int_a^a f(t)\,dt = 0.$$

DEFINITION $\int_b^a f(t)\,dt$ **[Switching the Limits of Integration]** More properities

Let f be integrable on the interval $[a, b]$ where $a < b$. Then we define

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

THEOREM 3 **Integrals over Subintervals** More properities

Assume that f is integrable on an interval I containing a, b and c. Then

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

☆ C does not need to be between a & b!

In general, if f is a continuous function on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx$$

represents the area of the region under the graph of f that lies *above* the x-axis between $x = a$ and $x = b$ ***minus*** the area of the region above the graph of f that lies *below* the x-axis between $x = a$ and $x = b$.

1.4 - The Average Value of a Function

DEFINITION **Average Value of f**

If f is continuous on $[a, b]$, the *average value of f on* $[a, b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(t)\,dt$$

THEOREM 4 **Average Value Theorem (Mean Value Theorem for Integrals)**

Assume that f is continuous on $[a, b]$.

Then there exists $a \le c \le b$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt$$

☆ Note that this theorem holds even if $b<a$ since $f(c) = \frac{1}{a-b}\int_b^a f(x)\,dx = \frac{1}{a-b}\left(-\int_a^b f(x)dx\right)$

$$= \frac{1}{b-a}\int_a^b f(x)dx.$$

1.5/6 - The Fundamental Theorem of Calculus

THEOREM 5 **Fundamental Theorem of Calculus (Part 1) [FTC1]**

Assume that f is continuous on an open interval I containing a point a. Let

one of the antiderivative

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x).$$

Equivalently,

$$\boxed{G'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

The collection of all antiderivatives of $f(x)$ is denoted by $\int f(x)\,dx$, and $\int f(x)dx = F(x) + C$ Indefinity integral

NOTE

If we use Leibniz notation for derivatives, the Fundamental Theorem of Calculus (Part 1) can be written as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This equation roughly states that if you first integrate f and then differentiate the result, you will return back to the original function f.

THEOREM 6 **Extended Version of the Fundamental Theorem of Calculus** / Leibniz Formula

Assume that f is continuous and that g and h are differentiable. Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)\,dt.$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x).$$

DEFINITION **Antiderivative**

Given a function f, an *antiderivative* is a function F such that

$$F'(x) = f(x).$$

If $F'(x) = f(x)$ for all x in an interval I, we say that F is an antiderivative for f on I.

THEOREM 7 **Power Rule for Antiderivatives**

If $\alpha \neq -1$, then

$$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C.$$

Integrand	Antiderivative
$f(x) = x^n$ where $n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$f(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln(\lvert x \rvert) + C$
$f(x) = e^x$	$\int e^x\,dx = e^x + C$
$f(x) = \sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$f(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$f(x) = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$f(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
$f(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
$f(x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$
$f(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$f(x) = a^x$ where $a > 0$ and $a \neq 1$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

THEOREM 8 **Fundamental Theorem of Calculus (Part 2) [FTC2]**

Assume that f is continuous and that F is any antiderivative of f.

Then

$$\int_a^b f(t)dt = F(b) - F(a).$$

if a function is continuous, then it has a antiderivative

We will now introduce the following notation to use in evaluating integrals. We write

$$\boxed{F(x)\Big|_a^b = F(b) - F(a)}$$

to indicate that the value of the antiderivative F evaluated at b minus the value of the antiderivative F evaluated at a.

1.7 - Change of Variables

THEOREM 9 **Change of Variables**

Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$, then

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

Trigonometric Substitution 2.1

Sometimes, Changing X into a trig function can simplify an integral:

Summary of Inverse Trigonometric Substitutions

So that the steps are invertable

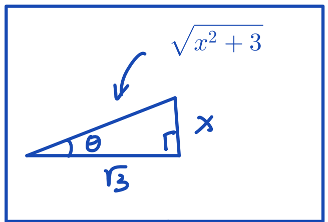
Class of Integrand	Integral	Trig Substitution	Trig Identity	Range for u
$\sqrt{a^2 - b^2x^2}$	$\int \sqrt{a^2 - b^2x^2}\, dx$	$bx = a\sin(u)$	$\sin^2(x) + \cos^2(x) = 1$	$-\frac{\pi}{2} < u < \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\int \sqrt{a^2 + b^2x^2}\, dx$	$bx = a\tan(u)$	$\sec^2(x) - 1 = \tan^2(x)$	$-\frac{\pi}{2} < u < \frac{\pi}{2}$
$\sqrt{b^2x^2 - a^2}$	$\int \sqrt{b^2x^2 - a^2}\, dx$	$bx = a\sec(u)$	$\sec^2(x) - 1 = \tan^2(x)$	$0 \leq u \leq \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3}{2}\pi$

Example 1:

$$\int \frac{x}{\sqrt{x^2+3}}dx$$

let $\underline{x = \sqrt{3}\tan\theta}, dx = \sqrt{3}\sec^2\theta d\theta$

$$\tan\theta = \frac{x}{\sqrt{3}} \rightarrow$$



$$\boxed{\sec\theta = \frac{\sqrt{x^2+3}}{\sqrt{3}}}$$

$$= \int \frac{\sqrt{3}\tan\theta \cdot \sqrt{3}\sec^2\theta}{\sqrt{\tan^2\theta + 1}}d\theta$$

$$= \frac{3}{\sqrt{3}}\int \frac{\tan\theta \cdot \sec^2\theta}{\sqrt{\tan^2\theta+1}}d\theta$$

$$= \frac{3}{\sqrt{3}}\int \frac{\tan\theta \cdot \sec^2\theta}{\sqrt{\sec^2\theta}}d\theta$$

$$= \frac{3}{\sqrt{3}}\int \frac{\tan\theta \cdot \sec^2\theta}{\sec\theta}d\theta$$

($\sec\theta > 0$ here)

$$= \frac{3}{\sqrt{3}}\int \tan\theta \cdot \sec\theta d\theta$$

$$= \sqrt{3}\sec\theta + c$$

(now we switch back to x)

$$= \sqrt{x^2+3} + c$$

Example II:

$$\int \frac{1}{x\sqrt{1-4x^2}}dx$$

$$= \int \frac{1}{x\sqrt{1-(2x)^2}}dx \longrightarrow$$

let $2x = \sin\theta$, so $2dx = \cos\theta d\theta \Rightarrow \boxed{dx = \frac{\cos\theta}{2}d\theta}$

$\boxed{x = \frac{\sin\theta}{2}}$

$$= \int \frac{1}{\frac{\sin\theta}{\cancel{2}}\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{\cancel{2}}dx$$

$$= \int \frac{\cancel{\cos\theta}}{\sin\theta\cancel{\sqrt{\cos^2\theta}}}d\theta$$

we have $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\cos\theta \geq 0$.

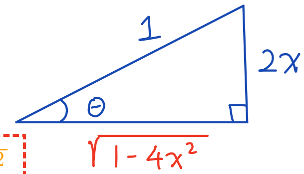
$$= \int \frac{1}{\sin\theta}d\theta$$

$$= \int \csc\theta d\theta$$

$$= -\ln(|\csc\theta + \cot\theta|) + c$$

By formula.

Now from $x = \frac{\sin\theta}{2}$, we could get $\sin\theta = \frac{2x}{1}$,



From triangle, we get $\boxed{\csc\theta = \frac{1}{\sin\theta} = \frac{1}{2x}}$, $\boxed{\cot\theta = \frac{\sqrt{1-4x^2}}{2x}}$

So, we get: $-\ln(|\csc\theta + \cot\theta|) + c = -\ln\left(\left|\frac{1}{2x} + \frac{\sqrt{1-4x^2}}{2x}\right|\right) + c$

Final answer

Example III:

$$\int \frac{1}{(x+1)^2\sqrt{x^2+2x-1}}dx$$

we want to complete $\sqrt{x^2+2x-1}$: $(x^2+2x)-1$

$= (x^2+2x+1-1)-1$

$= \boxed{(x+1)^2-2}$

$$= \int \frac{1}{(x+1)^2\sqrt{(x+1)^2-2}}dx$$

Let $(x+1) = \sqrt{2}\sec\theta$, $\Rightarrow dx = \sqrt{2}\sec\theta\tan\theta d\theta$

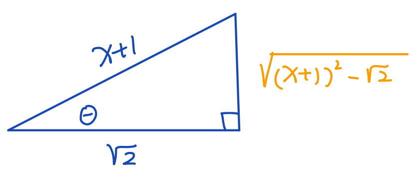
$$= \int \frac{\sqrt{2}\sec\theta\tan\theta}{2\sec^2\theta\sqrt{2\sec^2\theta - 2}}d\theta$$

$$= \frac{\cancel{\sqrt{2}}}{2\cdot\cancel{\sqrt{2}}}\int \frac{\cancel{\sec\theta\tan\theta}}{\sec^{\cancel{2}}\theta\cancel{\sqrt{\sec^2\theta - 1}}}d\theta$$

$\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$ we know its positive.

$$= \frac{1}{2}\int \frac{1}{\sec\theta}d\theta$$

$$= \frac{1}{2}\int \cos\theta d\theta$$



$\therefore \sin\theta = \frac{\sqrt{(x+1)^2 - \sqrt{2}}}{x+1}$

$$= \frac{\sin\theta}{2} + c$$

$$= \frac{\sqrt{(x+1)^2-2}}{2(x+1)} + c$$

Integration by Parts 2.2

DEFINITION **The Integration by Parts Formula** $\int u\,d\nu = uv - \int v\,du$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$ Reverse product rule

Strategy:

- When integrating the product of two functions, pick one to integrate (call it dv), and one to differentiate (call it u).
  - Pick dv to be the most difficult function you know how to integrate.
  - Pick u so that it gets simpler when differentiated.

Or, use the ILATE rule:

- Pick u to be the first function in the list, and dv be the rest:
  - I: inverse trig. functions
  - L: logarithmic functions
  - A: trig. functions
  - T: trig. functions
  - E: exponential functions

Example I:

$$\int_1^e x^2 \ln x\,dx$$ let $u = \ln x, dv = x^2 dx.$ so $du = \frac{1}{x}dx, v = \frac{x^3}{3}$

$$= \frac{x^3 \cdot \ln x}{3}\Big|_1^e - \int_1^e \frac{x^3}{3} \cdot \frac{1}{x}dx$$

$$= \frac{x^3 \cdot \ln x}{3}\Big|_1^e - \int_1^e \frac{x^2}{3}dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9}\Big|_1^e$$

$$= \left(\frac{e^3}{3}\ln(e) - \frac{e^3}{9}\right) - \left(\frac{1^3}{3}\ln(1) - \frac{1^3}{9}\right)$$

$$= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$

Example II:

$$\int e^x \sin x dx$$

Let $u = \sin x, dv = e^x dx$, so $du = \cos x dx, v = e^x$

Warning:

- if chose u to be the trig function, don't switch to the exponential function on the second attempt, it could undo your work.

$$= e^x \sin x - \int e^x \cos x dx$$

Let $x = \cos x, dv = e^x dx$, So $du = -\sin x, v = e^x$.

$$= e^x \sin x - \left[e^x \cos x - \int e^x \left(-\sin x \right) dx \right]$$

$$= e^x \sin x - e^x \cos x - \boxed{\int e^x \sin x dx}$$

The very same equation we started with, but with something else.

We've shown that: $I = e^x \sin x - e^x \cos x - 1$

$$2I = e^x \sin x - e^x \cos x$$

$$\boxed{I = \frac{e^x \sin x - e^x \cos x}{2} \quad + c}$$

Don't forget

We integrated this function without intergrating anything.

Example III:

$$\int x^3 e^{x^2} dx$$

$\left[\text{Let } u = x^2 \right.$, so $\left. dx = \frac{du}{2x} \right]$

We got a algebraic function, and a compositions of two functions. So we don't really have a exponential function by itself, so we need to be careful about what we are choosing here.
And in fact, we should not using integration by parts right away.

$$= \int x^{\overset{2}{\cancel{3}}} e^u \frac{du}{2\cancel{x}}$$

$$= \frac{1}{2} \int x^2 e^u dx$$

$$= \frac{1}{2} \int u \cdot e^u du$$

Now, integration by parts work. Let $u = u, du = du;$ $dv = e^u du, v = e^u$

$$= \frac{1}{2} \left[ue^u - \int e^u du \right]$$

$$= \frac{1}{2} ue^u - \frac{e^u}{2} + c$$

$$= \boxed{\frac{1}{2} x^2 e^{x^2} - \frac{e^{x^2}}{2} + C}$$

THEOREM 1

Integration by Parts

Assume that f and g are such that both f' and g' are continuous on an interval containing a and b. Then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx.$$

DEFINITION

Type I Partial Fraction Decomposition

Assume that

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials such that

1. $degree(p(x)) < degree(q(x)) = k$,
2. $q(x)$ can be factored into the product of linear terms each with distinct roots. That is
$$q(x) = a(x - a_1)(x - a_2)(x - a_3)\cdots(x - a_k)$$
where the a_i's are unique and none of the a_i's are roots of $p(x)$.

Then there exists constants $A_1, A_2, A_3, \cdots, A_k$ such that

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \frac{A_3}{x - a_3} + \cdots + \frac{A_k}{x - a_k}\right]$$

we say that f admits a *Type I Partial Fraction Decomposition*.

THEOREM 2

Integration of Type I Partial Fractions

Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type I Partial Fraction Decomposition of the form

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_k}{x - a_k}\right].$$

Then

$$\begin{aligned}\int f(x)\,dx &= \frac{1}{a}\left[\int \frac{A_1}{x - a_1}\,dx + \int \frac{A_2}{x - a_2}\,dx + \cdots + \int \frac{A_k}{x - a_k}\,dx\right]\\ &= \frac{1}{a}\left[A_1 \ln(|\, x - a_1 \,|) + A_2 \ln(|\, x - a_2 \,|)) + \cdots + A_k \ln(|\, x - a_k \,|)\right] + C\end{aligned}$$

DEFINITION

Type II Partial Fraction Decomposition

Assume that
$$f(x) = \frac{p(x)}{q(x)}$$
where $p(x)$ and $q(x)$ are polynomials such that

1. $degree(p(x)) < degree(q(x)) = k$,
2. $q(x)$ can be factored into the product of linear terms with non-distinct roots. That is
$$q(x) = a(x - a_1)^{m_1}(x - a_2)^{m_2}(x - a_3)^{m_3} \cdots (x - a_l)^{m_l}$$
where at least one of the m_j's is greater than 1.

We say that f admits a *Type II Partial Fraction Decomposition*.

In this case, the partial fraction decomposition can be built as follows.

Each expression $(x - a_j)^{m_j}$ in the factorization of $q(x)$ will contribute m_j terms to the decomposition, one for each power of $x - a_j$ from 1 to m_j, which when combined will be of the form

$$\frac{p(x)}{q(x)} = \sum_{j=1}^{l} \frac{A_{j,1}}{x - a_j} + \frac{A_{j,2}}{(x - a_j)^2} + \frac{A_{j,3}}{(x - a_j)^3} + \cdots + \frac{A_{j,m_j}}{(x - a_j)^{m_j}}.$$

The number m_j is called the *multiplicity* of the root a_j.

DEFINITION

Type III Partial Fraction Decomposition

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function with $degree(p(x)) < degree(q(x))$, but $q(x)$ does not factor into linear terms. We say that f admits a *Type III Partial Fraction Decomposition*.

In this case, the partial fraction decomposition can be built as follows:

Suppose that $q(x)$ has an irreducible factor $x^2 + bx + c$ with multiplicity m. Then this factor will contribute terms of the form

$$\frac{B_1 x + C_1}{x^2 + bx + c} + \frac{B_2 x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_m x + C_m}{(x^2 + bx + c)^m}$$

to the decomposition.

The linear terms are handled exactly as they were in the previous cases.

Partial Fractions 2.3

Partial fraction are useful for dividing a difficult integral to many simpler integrals, like: $\int \frac{p(x)}{q(x)} dx$

We will assume the degree of the denominator is larger than the degree of the numerator.
If not, use long division first.

Rules to break up fractions:

If the denominator has:	Then we write:
1): Distinct linear factors	one constant per factor
2): A repeated linear factor	one constant per power
3): Distinct irreducible quagratic factors	one linear term per factor
4): repeated irreducible quadratic factors	one linear term per power

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{c}{x-1}$$

one constant per power

Linear Term

$$\frac{x^3+x+7}{x^3(x+1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E\boxed{x+F}}{x^2+1}$$

As long as the power of x is higher in Denominator.

$$\frac{x^2+7}{(x-1)^3(x^2+3)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+3} + \frac{Fx+G}{(x^2+3)^2}$$

Example I:

$$\int \frac{5x+1}{(2x+1)(x-1)} dx$$

$$\Rightarrow \frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

We know $5x+1 = A(x+1) + B(2x+1)$

Fast way, But it don't work every time.
Idea: sub in some 'clever value' of x.

Let $x=1$: $6 = A(0) + B(3)$
$6 = 3B \Rightarrow B = 2.$

Let $x = -\frac{1}{2}$: $-\frac{5}{2} + 1 = A\left(-\frac{3}{2}\right) + B(0)$
$-\frac{3}{2} = -\frac{3}{2}A$
$A = 1$

$$= \int \frac{1}{2x+1} + \frac{2}{x-1} dx$$

$$= \frac{1}{2}\ln(2x+1) + 2ln(x-1) + C$$

$$\int \frac{1}{ax+b} = \frac{1}{a}\ln(ax+b)$$

Example II:

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$$

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$= A(x^3+x-x^2-1) + B(x^2+1) + (Cx+D)(x^2-2x+1)$$

$$= Ax^3 + Ax - Ax^2 - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$= (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + B - A + D$$

So we get: $\begin{cases} A + c = 0 \\ (-A+B-2C+D) = 1 \\ (A+C-2D) = -2 \\ (B-A+D) = -1. \end{cases}$

You can solve the system any way you want.

Simplification yields, $\begin{cases} D = 1 \\ A = 1 \\ C = -1 \\ B - 1 \end{cases}$

So we get :

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{-x+1}{x^2+1}dx$$

$$= \ln(x-1) + \frac{1}{x-1} + \underbrace{\int \frac{-x}{x^2+1}dx}_{(u = x^2+1)} + \underbrace{\int \frac{1}{x^2+1}dx}_{\text{arctan}}$$

$$= \ln(x-1) + \frac{1}{x-1} - \frac{1}{2}\cdot\ln\left(x^2+1\right) + \arctan x + C$$

$$\int \frac{-x}{x^2+1}dx$$

The Integration by Parts Formula $\int u\,dv = uv - \int v\,du$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$ Reverse product rule

let $u = x^2+1$, so $\frac{du}{dx} = 2x \Rightarrow du = 2x dx \Rightarrow dx = \frac{1}{2x}\cdot du$

So $\int \frac{-x}{x^2+1}dx = \int \frac{-x}{u}\cdot\frac{1}{2x}\cdot du$

$$= \int \frac{-1}{2u}du$$

$$= -\frac{1}{2}\int \frac{1}{u}du$$

$$= -\frac{1}{2}\ln(|u|) + C$$

$$= -\frac{1}{2}\ln\left(|x^2+1|\right) + C$$ sub back x.

Integrand	Antiderivative
$f(x) = x^n$ where $n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$f(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln(\lvert x \rvert) + C$
$f(x) = e^x$	$\int e^x\,dx = e^x + C$
$f(x) = \sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$
$f(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$
$f(x) = \sec^2(x)$	$\int \sec^2(x)\,dx = \tan(x) + C$
$f(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
$f(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
$f(x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$
$f(x) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$f(x) = a^x$ where $a > 0$ and $a \neq 1$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$

2.4 - Improper Integrals

So far, we have only examined Integrals of continuous, or at least bounded functions.

Let's see how to deal with a more general collection of functions!

In Particular, we will examine two types:

Type I: Infinite Intervals:
Integrals of the form:

- $\int_{-\infty}^{a} f(x)\,dx$
- $\int_{a}^{\infty} f(x)\,dx$
- $\int_{-\infty}^{\infty} f(x)\,dx$

Type II: Infinity Discontinuity:

Example: $\int_{-x}^{1} \frac{1}{x} dx \ldots$ There is a issue at $x = 0$.

<u>The idea in all cases is to replace the problematic point with a letter and take a limit.</u>

Type I

- We replace the infinite endpoint with a letter and take a limit.

- $\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to\infty} \int_{b}^{a} f(x)\,dx$
- $\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$
- $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b_1\to-\infty} \int_{b_1}^{0} f(x)\,dx + \lim_{b_2\to\infty} \int_{0}^{b_2} f(x)\,dx$

Don't use $\cancel{\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx}$

This is called the "Cavchy Principal Value" and it is something else!

We say that the integral <u>converges</u> if all of the limits exist (and are finite).

The Integral <u>diverges</u> if even one limit DNS (or is +- Infinity.)

Examples:
- Evaluate the following or show they diverge.

$$\int_{2}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_{2}^{b} \frac{1}{x^2} dx = \lim_{b\to\infty} -\frac{1}{x^2} \Big|_{2}^{b} = \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{2} = \boxed{\frac{1}{2}}$$

So it converges, and evaluate to 1/2.

Type II

- Suppose following integral converge: $\int_a^\infty f(x)\,dx$ $\int_a^\infty g(x)\,dx$

$1):$ $\int_a^\infty cf(x)\,dx$ converges for any $c \in \mathbb{R}$, and $\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$.

$2):$ $\int_a^\infty f(x)+g(x)\,dx$ converges, and $\int_a^\infty f(x)+g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$.

$3):$ If $f(x) \leq g(x)$ for all $x \geq a$, then $\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$.

$4):$ If $a < c < \infty$, then $\int_c^\infty f(x)\,dx$ converges, and $\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$.

Evaluating integrals in general is hard, and determining if an improper integral converges may be even harder! However, we do have a way of comparing a difficult integral to a simpler one (for example, a P-integral.)

DEFINITION **Type I Improper Integral**

Type I Improper Integral

1) Let f be integrable on $[a,b]$ for each $a \leq b$. We say that the *Type I Improper Integral*

$$\int_a^\infty f(x)\,dx$$

converges if

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

exists. In this case, we write

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Otherwise, we say that $\int_a^\infty f(x)\,dx$ *diverges*.

2) Let f be integrable on $[b,a]$ for each $b \leq a$. We say that the *Type I Improper Integral*

$$\int_{-\infty}^a f(x)\,dx$$

converges if

$$\lim_{b\to-\infty}\int_b^a f(x)\,dx$$

exists. In this case, we write

$$\int_{-\infty}^a f(x)\,dx = \lim_{b\to-\infty}\int_b^a f(x)\,dx.$$

Otherwise, we say that $\int_{-\infty}^a f(x)\,dx$ diverges.

3) Assume that f is integrable on $[a,b]$ for each $a,b \in \mathbb{R}$ with $a < b$. We say that the *Type I Improper Integral*

$$\int_{-\infty}^\infty f(x)\,dx$$

converges if both $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ converge for some $c \in \mathbb{R}$.
In this case, we write

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Otherwise, we say that $\int_{-\infty}^\infty f(x)\,dx$ diverges.

Absolute Convergence for Type I Improper Integrals

DEFINITION **Absolute Convergence for Type I Improper Integrals**

Let f be integrable on $[a,b)$ for all $b \geq a$. We say that the improper integral $\int_a^\infty f(x)\,dx$ *converges absolutely* if

$$\int_a^\infty |\,f(x)\,|\,dx$$

converges.

DEFINITION **The Gamma Function**

The Gamma Functions

For each $x \in \mathbb{R}$, define

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

The function Γ is called the Gamma function.

THEOREM 3 **p-Test for Type I Improper Integrals**

p-Test for Type I Improper Integral

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if and only if $p > 1$.

If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

Properties of Type I Improper Integrals

THEOREM 4 **Properties of Type I Improper Integrals**

Assume that $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge.

1. $\int_a^\infty cf(x)\,dx$ converges for each $c \in \mathbb{R}$ and
$$\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx.$$
2. $\int_a^\infty (f(x)+g(x))\,dx$ converges and
$$\int_a^\infty (f(x)+g(x))\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx.$$
3. If $f(x) \leq g(x)$ for all $a \leq x$, then
$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$
4. If $a < c < \infty$, then $\int_c^\infty f(x)\,dx$ converges and
$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

The Monotone Convergence Theorem for functions

THEOREM 5 **The Monotone Convergence Theorem for Functions**

Assume that f is non-decreasing on $[a,\infty)$.

1. If $\{f(x) \mid x \in [a,\infty)\}$ is bounded above, then $\lim_{x\to\infty} f(x)$ exists and
$$\lim_{x\to\infty} f(x) = L = lub(\{f(x) \mid x \in [a,\infty)\}).$$
2. If $\{f(x) \mid x \in [a,\infty)\}$ is not bounded above, then $\lim_{x\to\infty} f(x) = \infty$.

THEOREM 6 **Comparison Test for Type I Improper Integrals**

Assume that $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and that both f and g are continuous on $[a,\infty)$.

1. If $\int_a^\infty f(x)\,dx$ converges, then so does $\int_a^\infty g(x)\,dx$.
2. If $\int_a^\infty g(x)\,dx$ diverges, then so does $\int_a^\infty f(x)\,dx$.

THEOREM 7 **Absolute Convergence Theorem for Improper Integrals**

Let f be integrable on $[a,b]$ for all $b > a$. Then $|f|$ is also integrable on $[a,b]$ for all $b > a$. Moreover, if we assume that

$$\int_a^\infty |\,f(x)\,|\,dx$$

converges, then so does

$$\int_a^\infty f(x)\,dx.$$

In particular, if $0 \leq |f(x)| \leq g(x)$ for all $x \geq a$, both f and g are integrable on $[a,b]$ for all $b \geq a$, and if $\int_a^\infty g(x)\,dx$ converges, then so does

$$\int_a^\infty f(x)\,dx.$$

DEFINITION **Type II Improper Integral**

1) Let f be integrable on $[t, b]$ for every $t \in (a, b]$ with either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = -\infty$. We say that the *Type II Improper Integral*

$$\int_a^b f(x)\,dx$$

converges if

$$\lim_{t\to a^+} \int_t^b f(x)\,dx$$

exists. In this case, we write

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

Otherwise, we say that $\int_a^b f(x)\,dx$ *diverges*.

2) Let f be integrable on $[a, t]$ for every $t \in [a, b)$ with either $\lim_{x\to b^-} f(x) = \infty$ or $\lim_{x\to b^-} f(x) = -\infty$. We say that the *Type II Improper Integral*

$$\int_a^b f(x)\,dx$$

converges if

$$\lim_{t\to b^-} \int_a^t f(x)\,dx$$

exists. In this case, we write

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

Otherwise, we say that $\int_a^b f(x)\,dx$ *diverges*.

3) If f has an infinite discontinuity at $x = c$ where $a < c < b$, then we say that the *Type II Improper Integral*

$$\int_a^b f(x)\,dx$$

converges if both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. In this case, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If one or both of these integrals diverge, then we say that $\int_a^b f(x)\,dx$ *diverges*.

THEOREM 8 **p-Test for Type II Improper Integrals**

The improper integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges if and only if $p < 1$.

If $p < 1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}.$$

Area Between Curves

Let f and g be continuous on $[a, b]$. Let A be the region bounded by the graphs of f and g, the line $t = a$ and the line $t = b$. Then the area of region A is given by

$$A = \int_a^b |\, g(t) - f(t) \,|\, dt.$$

Week MT: February 22 - 26

Textbook Sections/Topics:

- Section 4.2 - Separable Differential Equations
- Section 4.3 - First-Order Linear Differential Equations
- Section 4.4 - Initial Value Problems
- Section 4.6 - Exponential Growth and Decay
- Section 4.7 - Newton's Law of Cooling
- Section 4.8 - Logistic Growth

4.2 - Separable Differential Equations

DEFINITION **Separable Differential Equation** Definition: Separable Differential Equation

A first-order differentiable equation is *separable* if there exists functions $f = f(x)$ and $g = g(y)$ such that the differentiable equation can be written in the form

$$y' = f(x)g(y).$$

EXAMPLE 3 Consider the following differentiable equations:

i) $y' = xy^2$ is separable. In this case, $f(x) = x$ and $g(y) = y^2$.

ii) $y' = y$ is separable. In this case, $f(x) = 1$ and $g(y) = y$.

iii) $y' = \cos(xy)$ is not separable since it can not be written in the form $y' = f(x)g(y)$. ◂

DEFINITION **Constant (Equilibrium) Solution to a Separable Differential Equation** Definition: Constant (Equilibrium) Solution to a Separable Differential Equation

If

$$y' = f(x)g(y)$$

is a separable differential equation and if $y_0 \in \mathbb{R}$ is such that $g(y_0) = 0$, then

$$\phi(x) = y_0$$

is called a *constant* or *equilibrium solution* to the differential equation.

Strategy [Solving Separable Differential Equations]

Strategy [Solving Separable Differential Equations]

Solving the separable differential equation

$$y' = f(x)g(y)$$

consists of 4 steps.

Step 1: Determine whether the DE is separable. You may have to factor the DE to identify $f(x)$ and $g(y)$.

Step 2: Determine the *constant solution(s)* by finding all the values y_0 such that $g(y_0) = 0$. For each such y_0, the constant function

$$y = y(x) = y_0$$

is a solution.

Step 3: If $g(y) \neq 0$, integrate both sides of the following equation

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

to solve the differential equation implicitly.

Step 4: Solve the implicit equation from Step 3 explicitly for y in terms of x.

Step 5: **[Optional]** Check your solution by differentiating y to deterimine if this derivative is equal to the original DE y'.

Solving Separable Differential Equations

There is a simple process to follow to find the solutions to a separable differential equation $y' = f(x)g(y)$. The steps are:

Step 1: Identify $f(x)$ and $g(y)$

Step 2: Find all constant (equilibrium) solutions

Step 3: Find the implicit solution

Step 4: Find the explicit solutions

Step 1: Identify $f(x)$ and $g(y)$

Often when you are presented with a differential equation, it will not be obvious whether the DE is separable. You may have to factor the differential equation in order to identify $f(x)$ and $g(y)$.

Step 2: Find all constant (equilibrium) solutions

Let

$$y' = f(x)g(y)$$

be a separable DE.

Suppose that $g(y_0) = 0$ for some y_0. Then the constant function

$$y = \varphi(x) = y_0$$

is a solution to the separable differential equation since

$$\varphi'(x) = 0 = f(x)g(y_0) = f(x)g(\varphi(x))$$

for every x.

Step 3: Find the implicit solution

If $y' = f(x)g(y)$ is a separable differential equation, when $g(y) \neq 0$ we can divide by $g(y)$ to get

$$\frac{y'}{g(y)} = f(x).$$

Integrating both sides with respect to x gives

$$\int \frac{y'}{g(y)}\,dx = \int f(x)\,dx.$$

However, if we note that $y = y(x)$, we can apply the Change of Variables theorem to the left-hand integral to get

$$\begin{aligned}\int \frac{y'}{g(y)}\,dx &= \int \frac{1}{g(y(x))}y'(x)\,dx \\ &= \int \frac{1}{g(y)}dy\end{aligned}$$

This gives us the formula

$$\boxed{\int \frac{1}{g(y)}dy = \int f(x)\,dx}$$

Evaluating these integrals gives us an *implicit solution* to the differential equation of the form

$$G(y) = F(x) + C$$

where C is an arbitrary real constant.

This step will be successful only if we are able to evaluate $\int \frac{1}{g(y)}dy$ and $\int f(x)\,dx$.

Step 4: Find the explicit solutions

Try to solve the implicit equation

$$G(y) = F(x) + C$$

for y in terms of x. This will be the explicit solution to the differential equation. Unfortunately, it is not always easy to solve this equation for y in terms of x.

4.3 - First-Order Linear Differential Equations

Definition: First-Order Linear Differentiable Equations [FOLDE]

DEFINITION **First-Order Linear Differentiable Equations [FOLDE]**

A first-order differential equation is said to be *linear* if it can be written in the form

$$y' = f(x)y + g(x).$$

EXAMPLE 8 Consider the following differential equations:

i) The separable differential equation

$$y' = 3x(y-1)$$

may be rewritten as

$$y' = 3xy - 3x$$

so it is also linear.

ii) The differentiable equation

$$y' = x^2y^3$$

is not linear since the term y^3 is of third degree.

◄

Strategy [Solving First-Order Differential Equations]

Strategy [Solving First-Order Linear Differential Equations]

Solving the first-order linear differential equation

$$y' = f(x)y + g(x)$$

consists of 3 steps.

Step 1: Determine whether the DE is linear. Write the equation in the form

$$y' - f(x)y = g(x)$$

and identify $f(x)$ and $g(x)$.

Step 2: Calculate the integrating factor $I(x)$ with $I(x) \neq 0$. Solve for I by using

$$I = e^{-\int f(x)\,dx}$$

Step 3: Since $I(x) \neq 0$, the solution is

$$y = \frac{\int g(x)I(x)\,dx}{I(x)}$$

Step 4: **[Optional]** Check your solution by differentiating y.

Theorem: Solving First-Order Linear Differential Equations

THEOREM 1 **Solving First-Order Linear Differential Equations**

Let f and g be continuous and let

$$y' = f(x)y + g(x)$$

be a first-order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)\,dx}{I(x)}$$

where $I(x) = e^{-\int f(x)\,dx}$.

Note: In theory, the method we have just outlined provides us with a means of solving all first-order linear differential equations. However, in practice this only works provided that we can perform the required integrations.

EXAMPLE 9 Solve the first-order linear differential equation

$$y' = 3xy - 3x.$$

The first step is to rewrite the differential equation so that "$g(x)$" is alone on the right-hand side of the equation,

$$y' - 3xy = -3x.$$

The next step is to multiply both sides of the equation by a *nonzero* function $I = I(x)$ to get

$$Iy' - 3xIy = -3xI \qquad (1)$$

The goal is to find the nonzero function $I = I(x)$ such that if we differentiate $I(x)y(x)$ we will get the left-hand side of equation (1). That is,

$$\frac{d}{dx}(I(x)y(x)) = Iy' - 3xIy$$

Using the Product Rule we see that

$$\frac{d}{dx}(I(x)y(x)) = Iy' + I'y$$

so we require that

$$Iy' + I'y = Iy' - 3xIy \qquad (2)$$

A close look at equation (2) shows us that we need

$$I' = -3xI \qquad (3)$$

Equation (3) is a separable differential equation which we know how to solve.

Since the only constant solution is $I = I(x) = 0$ and we require a nonzero function, we proceed to Step 2 of the algorithm for solving separable equations.

Write

$$\int \frac{1}{I}\,dI = \int (-3x)\,dx.$$

This gives

$$\ln(|\,I\,|) = -\frac{3}{2}x^2 + C.$$

Exponentiating shows that

$$|\,I\,| = C_1 e^{-\frac{3}{2}x^2}$$

where $C_1 = e^C > 0$ and hence that

$$I = C_2 e^{-\frac{3}{2}x^2}$$

with $C_2 \neq 0$.

We only require one such function, so choose $C_2 = 1$. Then

$$I = I(x) = e^{-\frac{3}{2}x^2}.$$

With this choice of I we now have an equation of the form

$$\frac{d}{dx}(I(x)y(x)) = -3xI$$

where $I = I(x) = e^{\frac{-3}{2}x^2}$.

Integrating both sides of this equation gives us

$$\begin{aligned} I(x)y &= \int \left(\frac{d}{dx}(I(x)y(x))\right) dx \\ &= \int -3xI(x)\,dx \\ &= \int -3xe^{-\frac{3}{2}x^2}\,dx \end{aligned}$$

Let $u = \frac{-3}{2}x^2$ to get that $du = -3xdx$ so $dx = \frac{du}{-3x}$ which gives

$$\begin{aligned} \int (-3xI(x))\,dx &= \int -3xe^{\frac{-3}{2}x^2}\,dx \\ &= \int e^u\,du \\ &= e^u + C \\ &= e^{\frac{-3}{2}x^2} + C \end{aligned}$$

This means

$$I(x)y = e^{\frac{-3}{2}x^2} + C.$$

Solving for y gives us

$$\begin{aligned} y &= \frac{e^{\frac{-3}{2}x^2} + C}{I(x)} \\ &= \frac{e^{\frac{-3}{2}x^2} + C}{e^{\frac{-3}{2}x^2}} \\ &= 1 + Ce^{\frac{3}{2}x^2} \end{aligned}$$

where C is an arbitrary constant.

Finally, we can verify this answer by differentiating y to get

$$\begin{aligned} y' &= Ce^{\frac{3}{2}x^2}(3x) \\ &= (3x)(y-1) \\ &= 3xy - 3x \end{aligned}$$

which, as we expected, is the original DE that we were trying to solve. ◄

4.4 - Initial Value Problems

Theorem: Existence and Uniqueness Theorem for First-Order Linear Differential Equations

THEOREM 2 **Existence and Uniqueness Theorem for First-Order Linear Differential Equations**

Assume that f and g are continuous functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem

$$y' = f(x)y + g(x)$$
$$y(x_0) = y_0$$

has exactly *one* solution $y = \varphi(x)$ on the interval I.

EXAMPLE 11 Solve the initial value problem

$$y' = xy$$

with $y(0) = 1$.

Observe that this differential equation is linear since it takes the form $y' = f(x)y + g(x)$ where $f(x) = x$ and $g(x) = 0$, so the previous theorem tells us that there will be a unique solution. However, this differential equation is also separable since it can be written in the form $y' = f(x)g(y)$ with $f(x) = x$ and $g(y) = y$, so we can use the method developed for separable equations to find the solution.

The only constant solution is $y = y(x) = 0$ which does not satisfy the initial conditions. Hence we have

$$\int \frac{1}{y}\,dy = \int x\,dx.$$

This shows that

$$\ln(|\,y\,|) = \frac{x^2}{2} + C$$

so

$$y = C_1 e^{\frac{x^2}{2}}.$$

We also have that

$$1 = y(0) = C_1 e^{\frac{0^2}{2}} = C_1 e^0 = C_1.$$

Therefore $y = e^{\frac{x^2}{2}}$ is the unique solution to this initial value problem. ◀

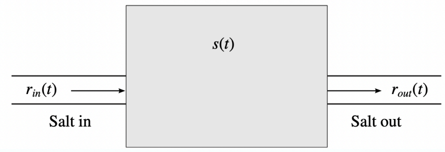
EXAMPLE 12 **A Mixing Problem**

Assume that a brine containing $30g$ of salt per litre of water is pumped into a $1000L$ tank at a rate of 1 litre per second. The tank initially contains $1000L$ of fresh water. It also contains a device that thoroughly mixes its contents. The resulting solution is simultaneously drained from the tank at a rate of 1 litre per second.

Problem: How much salt will be in the tank at any given time?

Let $s(t)$ denote the amount of salt in the tank at time t. Then $s'(t)$ is the difference between the rate at which salt is entering the tank (in the brine) and the rate at which salt is leaving the tank (in the discharge). Label these $r_{\text{in}}(t)$ and $r_{\text{out}}(t)$, respectively. That is,

$$s'(t) = r_{\text{in}}(t) - r_{\text{out}}(t).$$



To find $r_{\text{in}}(t)$ we note that the concentration of salt in the brine entering the tank is constant at $30g$ per litre. The flow rate is $1L$ per second and the rate at which the salt is entering the tank is the product of the concentration and the flow rate. Hence

$$r_{\text{in}}(t) = 30\frac{g}{L} \times 1\frac{L}{s} = 30\frac{g}{s}$$

and so the rate at which salt is entering the tank is 30 grams per second.

Calculating $r_{\text{out}}(t)$ is similar. It is the concentration of the discharge times the rate of flow. The rate of flow is again $1L$ per second but this time the concentration is not constant. In fact the concentration of the discharge is the same as that of the tank. Since the concentration of salt in the tank is $\frac{s(t)}{1000}$, we get

$$r_{\text{out}}(t) = \frac{s(t)}{1000} \times 1 = \frac{s(t)}{1000}$$

grams per second. It follows that

$$s'(t) = 30 - \frac{s(t)}{1000}.$$

This is a first-order linear differential equation with $f(t) = -\frac{1}{1000}$ and $g(t) = 30$. (Note: It is also a separable DE). To solve the equation as a FOLDE, the integrating factor $I(t)$ is

$$I(t) = e^{-\int \frac{-1}{1000}\,dt} = e^{\frac{t}{1000}}.$$

Using the FOLDE formula

$$y = \frac{\int g(t)I(t)dt}{I(t)}$$

gives us

$$\begin{aligned} s(t) &= \frac{\int 30e^{\frac{t}{1000}}\,dt}{e^{\frac{t}{1000}}} \\ &= \frac{30000e^{\frac{t}{1000}} + C}{e^{\frac{t}{1000}}} \\ &= 30000 + Ce^{-\frac{t}{1000}} \end{aligned}$$

Since $s(0) = 0$, we get

$$0 = 30000 + Ce^0 = 30000 + C$$

and hence

$$C = -30000.$$

Therefore, at any given time

$$s(t) = 30000 - 30000e^{-\frac{t}{1000}}$$

grams.

Finally, since $\lim_{x\to\infty} e^{-x} \to 0$, observe that

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} 30000 - 30000e^{-\frac{t}{1000}} = 30000$$

grams. This means that if the system was allowed to continue indefinitely, the amount of salt in the tank would approach 30000 grams. At that level, the concentration in the $1000L$ tank would be 30 grams per litre, which would be the same as the inflow rate. Therefore, the system is moving towards a stable equilibrium.

◀

4.6 - Exponential Growth and Decay

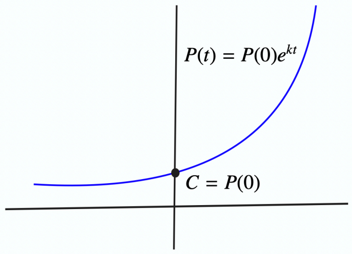
It is known that a population of bacteria in an environment with unlimited resources grows at a rate that is proportional to the size of the population. Therefore, if $P(t)$ represents the size of the population at time t, there is a constant k such that

$$P' = kP.$$

The general solution to this differential equation is given by

$$\boxed{P(t) = Ce^{kt}}$$

where $C = P(0)$ represents the initial population.



Exponential Growth

From the shape of the graph, it makes sense when we say that the bacteria population exhibits *exponential growth*.

Physical considerations generally limit the possible solutions to the equation. In the case of the bacteria population we will see that if we know the initial population as well as the size of the population at a one other fixed time, then the exact population function can be determined.

EXAMPLE 13 At time $t = 0$, a bacteria colony's population is estimated to be 7.5×10^5. One hour later, at $t = 1$, the population has doubled to 1.5×10^6. How long will it take until the population reaches 10^7?

Let $P(t)$ represent the size of the population at time t. We know that there is a constant k such that

$$P' = kP$$

so

$$P(t) = Ce^{kt}$$

and $C = P(0) = 7.5 \times 10^5$.

We also know that

$$1.5 \times 10^6 = P(1) = 7.5 \times 10^5 e^{k(1)}.$$

Therefore

$$e^k = \frac{1.5 \times 10^6}{7.5 \times 10^5} = 2.$$

To find k, take the natural logarithm of both sides of the equation to get

$$k = \ln(2).$$

This tells us that the population function is

$$P(t) = 7.5 \times 10^5 e^{(\ln(2))t}.$$

Now that we know the general formula for $P(t)$, to answer the original question we need to find t_0 such that

$$P(t_0) = 7.5 \times 10^5 e^{(\ln(2))t_0} = 10^7.$$

Therefore,

$$e^{(\ln(2))t_0} = \frac{10^7}{7.5 \times 10^5}$$

so

$$(\ln(2))t_0 = \ln\left(\frac{10^7}{7.5 \times 10^5}\right)$$

and

$$t_0 = \frac{\ln\left(\frac{10^7}{7.5 \times 10^5}\right)}{\ln(2)} \cong 3.74 \text{ hours.}$$

◀

There are many other real world phenomena that behave in a manner similar to the growth of a bacteria population. In other cases, rather than exponential growth, we have exponential *decay*. For example, the rate at which radioactive material breaks down is proportional to the mass of material present.

Let $m(t)$ denote the mass of a certain radioactive material at time t. Then there is a constant k such that

$$\frac{dm}{dt} = m' = km.$$

We have

$$m(t) = Ce^{kt}$$

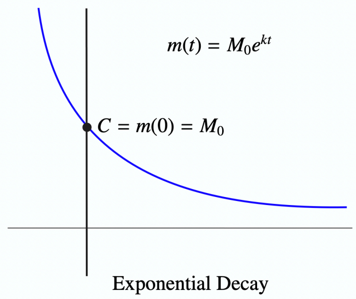
where $C = m(0) = M_0$ is the initial mass of the material. Therefore,

$$m(t) = M_0 e^{kt}.$$

Since the amount of material is decreasing, $m'(t) < 0$. But

$$m'(t) = km(t)$$

and $m(t) > 0$ so it follows that $k < 0$. Therefore, the graph of $m(t)$ appears as follows:



Exponential Decay

In particular, notice that

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} M_0 e^{kt} = 0$$

since $k < 0$.

We call such a process *exponential decay*.

All radioactive materials have associated with them a quantity t_h known as the *half-life* of the material. This is the amount of time it would take for one-half of the material to decay. The half-life is a fundamental characteristic of the material.

Mathematically, if

$$m(t) = M_0 e^{kt}$$

then t_h is the time at which

$$m(t_h) = M_0 e^{kt_h} = \frac{M_0}{2}.$$

Dividing by M_0 shows that

$$e^{kt_h} = \frac{1}{2}$$

and hence that

$$kt_h = \ln\left(\frac{1}{2}\right) = -\ln(2).$$

Therefore, the half-life is given by the formula

$$\boxed{t_h = \frac{-\ln(2)}{k}.}$$ Half-life's formula

In particular, this shows that the half-life of a material is independent of the original mass.

EXAMPLE 14 **Carbon Dating**

All living organisms contain a small amount of radioactive carbon-14. Moreover, each type of organism has a particular equilibrium ratio of carbon-14 compared to the stable isotope carbon-12.

When an organism dies the equilibrium is no longer maintained since the radioactive carbon-14 slowly breaks down into carbon-12. It is also known that carbon-14 breaks down at a rate of 1 part in 8000 per year. This means that after 1 year an initial quantity of 8000 particles will be reduced to 7999. Hence

$$7999 = m(1) = 8000e^{k(1)}$$

so that

$$k = \ln\left(\frac{7999}{8000}\right).$$

Problem 1: Find the half-life of carbon-14.

From the previous discussion, we know that

$$t_h = \frac{-\ln(2)}{k}$$

4.7 - Newton's Law of Cooling

Newton's law of cooling states that an object will cool (or warm) at a rate that is proportional to the difference between the temperature of the object and the ambient temperature T_a of its surroundings. Therefore, if $T(t)$ denotes the temperature of an object at time t, then there is a constant k such that

$$T' = k(T - T_a).$$

If $D = D(t) = T(t) - T_a$, then

$$D' = T' = kD$$

so D satisfies the equation for exponential growth (or decay). We know

$$D = Ce^{kt}.$$

It follows that

$$T(t) = Ce^{kt} + T_a$$

where $C = D(0) = T_0 - T_a$ and $T_0 = T(0)$.

Therefore,

$$\boxed{T(t) = (T_0 - T_a)e^{kt} + T_a.}$$

T_0: Intitial temperature
T_a: Ambient temperature
t: time
k: the constant

There are three possible cases.

1. $T_0 > T_a$.
   Physically, this means that the object is originally at a temperature that is greater than the ambient temperature. This means that the object will be *cooling*.
   Since $T(t)$ is decreasing
   $$T' = k(T - T_a) < 0.$$
   However, $T > T_a$, so that $k < 0$.
2. $T_0 < T_a$.
   In this case, the object is originally at a temperature that is lower than the ambient temperature. Therefore, the object will be *warming*.
   This time $T(t)$ is increasing so
   $$T' = k(T - T_a) > 0.$$
   Since $T < T_a$, it follows again that $k < 0$.
3. $T_0 = T_a$.
   Then
   $$T' = k(T - T_a) = 0$$
   Equilibrium state
   so the temperature remains constant. We call this the *equilibrium state*.

Notice that in all three cases,

$$\lim_{t\to\infty} T(t) = T_a.$$

Regardless of the initial starting point, if a process always moves towards a particular equilibrium value, we call this value a *stable equilibrium*.

EXAMPLE 15 A cup of boiling water at $100°C$ is allowed to cool in a room where the ambient temperature is $20°C$. If after 10 minutes the water has cooled to $70°C$, what will be the temperature after the water has cooled for 25 minutes?

Let $T(t)$ denote the temperature of the water at time t minutes after cooling commences. The initial temperature is $T_0 = 100°C$ and the ambient temperature is $T_a = 20°C$. Newton's Law of Cooling shows that there is a constant $k < 0$ such that

$$\begin{aligned} T(t) &= (T_0 - T_a)e^{kt} + T_a \\ &= (100 - 20)e^{kt} + 20 \\ &= 80e^{kt} + 20 \end{aligned}$$

The next step is to determine k. Note that

$$70 = T(10) = 80e^{k(10)} + 20$$

so

$$50 = 80e^{10k}.$$

Hence,

$$10k = \ln\left(\frac{50}{80}\right)$$

and

$$\begin{aligned} k &= \frac{\ln\left(\frac{50}{80}\right)}{10} \\ &= -0.047 \end{aligned}$$

We can now evaluate $T(25)$ to get that the temperature after 25 minutes is

$$\begin{aligned} T(25) &= 80e^{-0.047(25)} + 20 \\ &= 44.71 \end{aligned}$$

degrees Celsius.

$$\begin{aligned} &= \frac{-\ln(2)}{\ln\left(\frac{7999}{8000}\right)} \\ &\cong 5544.83 \text{ years} \end{aligned}$$

Problem 2: After a fossil was found research showed that the amount of carbon-14 was 23% of the amount that would have been present at the time of death. How old was the fossil?

Let M_0 be the expected amount of carbon-14 in the fossil and let t_o be the age of the fossil. Then the research shows that

$$(0.23)M_0 = m(t_0) = M_0e^{kt_0}.$$

We must solve this equation for t_0. The first step is to recognize that

$$e^{kt_0} = \frac{(0.23)M_0}{M_0} = 0.23$$

This shows that we did not need to find the quantity M_0 explicitly to solve this question.

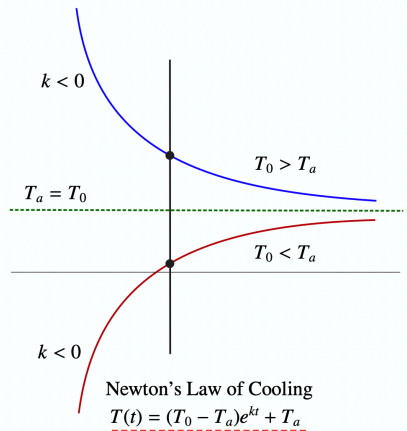
Taking the natural logarithm of both sides of the equation gives

$$kt_0 = \ln(0.23)$$

and hence that

$$\begin{aligned} t_0 &= \frac{\ln(0.23)}{k} \\ &= \frac{\ln(0.23)}{\ln\left(\frac{7999}{8000}\right)} \\ &= 11\,756 \text{ years} \end{aligned}$$

The diagram summarizes the possible graphs of the temperature function.



Newton's Law of Cooling
$T(t) = (T_0 - T_a)e^{kt} + T_a$

4.8 - Logistic Growth

We have seen that a population with unlimited resources grows at a rate that is proportional to its size. This leads to the differential equation

$$P' = kP.$$

However, the assumption that resources will be unlimited is usually unrealistic. More likely, there is a maximum population M that the surrounding environment can sustain. This means that as the population $P(t)$ approaches M, resources will become more scarce and the growth rate of the population will slow. On the other hand, when the population is small in comparison to the maximum population possible, the growth rate will be similar to that of the unrestricted case since there will be little resource pressure. It is known that such a population satisfies a differential equation of the form

$$\boxed{P' = kP(M - P).}$$

M: max population
P: current population

This equation means that the rate of growth is proportional to the product of the current population and the difference from the maximum sustainable population.

Populations of this type are said to satisfy *logistic growth* and the differential equation

$$\boxed{y' = ky(M - y)}$$

Logistic equation

is called the *logistic equation*.

The logistic equation need not only model a population. However, in the special case where we are trying to describe the behavior of a population, we have the additional constraint that $P(t) > 0$.

Let $P_0 = P(0)$ be the initial population at the beginning of a study.

Observe that if the initial population is smaller than M, then the population will be growing. This means that we would have

$$0 < P' = kP(M - P)$$

since both P and $M - P$ are positive. As such, we would expect that $k > 0$.

However, if the initial population exceeds the maximum sustainable population, then the population would decrease so

$$0 > P' = kP(M - P)$$

and again we would have $k > 0$ since $P > 0$ and $M - P < 0$.

A third possible case occurs when the initial population is already at the maximum. In this case,

$$P' = kP(M - P) = 0$$

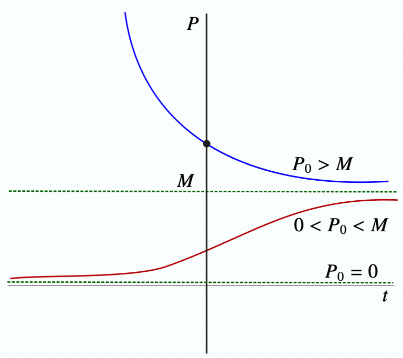
so the population would remain constant. This shows that $P(t) = M$ is an equilibrium solution. Equilibrium solution

The last case we will consider occurs when $P_0 = 0$. In this case, we have that

$$P' = kP(M - P) = 0$$

which makes sense since there are no parents to produce offspring. Therefore, $P(t) = 0$ is also an equilibrium, but its nature is quite different than that of the equilibrium at $P(t) = M$.

It follows that in all cases, we may assume that $P(t) > 0$ for all t so that the possible solutions look as follows:



Logistic Growth

You will notice that as long as $P_0 \neq 0$ we have

$$\boxed{\lim_{t\to\infty} P(t) = M.}$$

This means that $P(t) = M$ is a *stable* equilibrium. However, since we will never move towards an equilibrium of $P(t) = 0$ once there is a nonzero population, $P(t) = 0$ is called an ***unstable equilibrium***.

So far, we have presented a qualitative solution to the logistic growth problem. However, since the equation is separable, we can try to solve it algebraically. We have already observed that $P(x) = 0$ and $P(x) = M$ are the constant solutions. We can then try to solve

Strategy [Solving Separable Differential Equations]: Step 3

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt = kt + C_1$$

To evaluate $\int \frac{1}{P(M-P)}\,dP$ we use partial fractions.

The constants A and B are such that

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

or

$$1 = A(M-P) + B(P).$$

Letting $P = 0$ gives

$$1 = A(M)$$

so

$$A = \frac{1}{M}.$$

Letting $P = M$, we get

$$1 = B(M)$$

and again

$$B = \frac{1}{M}.$$

Therefore

$$\frac{1}{P(M-P)} = \frac{1}{M}\left[\frac{1}{P} + \frac{1}{M-P}\right].$$

It follows that

$$\begin{aligned}\int \frac{1}{P(M-P)}\,dP &= \frac{1}{M}\left[\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP\right]\\ &= \frac{1}{M}[\ln(|P|) - \ln(|M-P|)] + C_2\\ &= \frac{1}{M}\ln\left(\frac{|P|}{|M-P|}\right) + C_2\end{aligned}$$

We now have that

$$\frac{1}{M}\ln\left(\frac{|P(t)|}{|M-P(t)|}\right) + C_2 = kt + C_1.$$

Therefore,

$$\ln\left(\frac{|P(t)|}{|M-P(t)|}\right) = Mkt + C_3$$

where C_3 is arbitrary.

This shows that

$$\frac{|P(t)|}{|M-P(t)|} = Ce^{Mkt}$$

where $C = e^{C_3} > 0$.

There are two cases to consider.

Case 1: Assume that $0 < P(t) < M$. Then

$$\frac{|P(t)|}{|M-P(t)|} = \frac{P(t)}{M-P(t)} = Ce^{Mkt}.$$

Solving for $P(t)$ would give

$$\begin{aligned}P(t) &= (M - P(t))Ce^{Mkt}\\ &= MCe^{Mkt} - P(t)Ce^{Mkt}\end{aligned}$$

so that

$$P(t) + P(t)Ce^{Mkt} = MCe^{Mkt}.$$

We then have

$$P(t)(1 + Ce^{Mkt}) = MCe^{Mkt}$$

and finally that

$$\begin{aligned}P(t) &= \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}\\ &\boxed{= M\frac{Ce^{Mkt}}{1 + Ce^{Mkt}}}\end{aligned}$$

There are two important observations we can make about this solution.

(a) Since $C > 0$, the denominator is never 0 so the function $P(t)$ is continuous and

$$0 < \frac{Ce^{Mkt}}{1 + Ce^{Mkt}} < 1$$

so that

$$0 < P(t) < M$$

which agrees with our assumption. *

* (b) Since $k > 0$, we have that

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} M\frac{Ce^{Mkt}}{1+Ce^{Mkt}} \\ &= M\lim_{t\to\infty}\frac{Ce^{Mkt}}{1+Ce^{Mkt}} \\ &= M\end{aligned}$$

and

$$\lim_{t\to-\infty} P(t) = \lim_{t\to-\infty} M\frac{Ce^{Mkt}}{1+Ce^{Mkt}} = 0.$$

This shows that the population would eventually approach the maximum population M but if you went back in time far enough, the population would be near 0. Both of these limits are consistent with our expectations.

If $t = 0$, then

$$P_0 = P(0) = M\frac{Ce^0}{1+Ce^0} = M\frac{C}{1+C}.$$

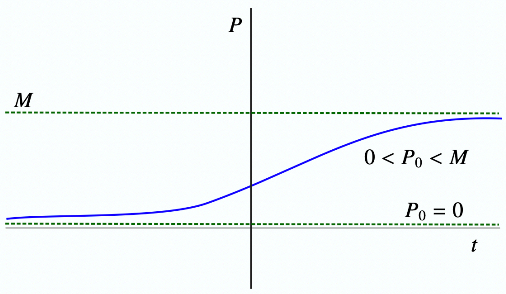
Solving for C yields

$$P_0(1+C) = MC$$
$$P_0 + P_0C = MC$$
$$P_0 = (M - P_0)C$$

and finally that

$$\boxed{C = \frac{P_0}{M - P_0}.}$$

The graph of the function $P(t) = M\dfrac{Ce^{Mkt}}{1+Ce^{Mkt}}$ looks as follows:



Logistic Growth

Case 2: If $P(0) > M$, then

$$\frac{|P(t)|}{|M - P(t)|} = -\frac{P(t)}{M - P(t)} = \frac{P}{P - M} = Ce^{Mkt}.$$

Proceeding in a manner similar to the previous case, we get that there exists a positive constant C such that

$$\boxed{P(t) = M\frac{Ce^{Mkt}}{Ce^{Mkt} - 1}.}$$

Notice that this function has a vertical asymptote when the denominator

$$Ce^{Mkt} - 1 = 0.$$

Moreover, the function is only positive if

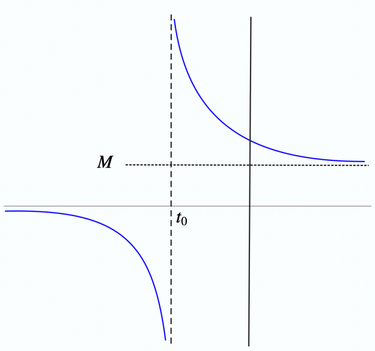
$$Ce^{Mkt} > 1$$

or equivalently if

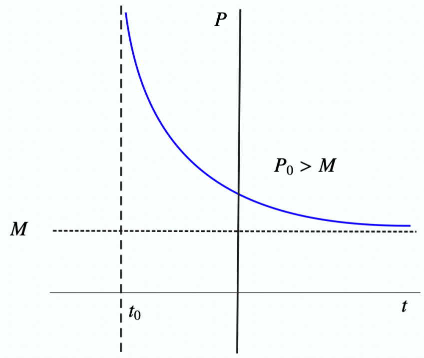
$$e^{Mkt} > \frac{1}{C}.$$

The use of some algebra shows that this happens if and only if $t > \dfrac{\ln\left(\frac{1}{C}\right)}{Mk} = t_0$.

If we ignore the fact that population must be positive, the graph of the solution function $\boxed{P(t) = M\dfrac{Ce^{Mkt}}{Ce^{Mkt} - 1}}$ appears as follows:



Since we are looking for a population function and so we require $P(t) \geq 0$, we will only consider values of t which exceed t_0. Therefore, the graph of the population function is:



It is still true that

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} M\frac{Ce^{Mkt}}{Ce^{Mkt} - 1} = M.$$

and

$$\lim_{t\to t_0^+} P(t) = \infty.$$

EXAMPLE 17 A rumor is circulating around a university campus. A survey revealed that at one point only 5% of the students in the school were aware of the rumor. However, since news on campus spreads quickly, after 10 hours the rumor is known by 10% of the student body. How long will it take until 30% of the students are aware of the rumor?

Let $r(t)$ be the fraction of the student body at time t that have heard this rumor. Then $0 \leq r(t) \leq 1$.

Experiments have shown that the rate at which a rumor spreads through a population is proportional to the product of the fraction of the population that have heard the rumor and the fraction that have not. Therefore, there is a constant k such that

$$r' = kr(1 - r)$$

and so this is a logistic growth model with $M = 1$. It follows that there is a positive constant C such that

$$\boxed{r(t) = \frac{Ce^{kt}}{1+Ce^{kt}}.}$$ $0 < r(t) < M\ (1)$

We know that at $r(0) = 0.05$ so

$$C = \frac{r(0)}{1 - r(0)} = \frac{0.05}{0.95} = 0.0526315$$

and hence that

$$\begin{aligned}0.1 &= r(10) \\ &= \frac{\frac{0.05}{0.95}e^{10k}}{1 + \frac{0.05}{0.95}e^{10k}}\end{aligned}$$

Therefore

$$0.1 + \frac{0.005}{0.95}e^{10k} = \frac{0.05}{0.95}e^{10k}$$

and

$$0.1 = \frac{0.045}{0.95}e^{10k}.$$

This gives

$$e^{10k} = \frac{0.095}{0.045}$$

and

$$k = \frac{\ln\left(\frac{0.095}{0.045}\right)}{10} = 0.07472$$

Finally, we want to find t_0 such that

$$0.3 = \frac{Ce^{kt_0}}{1+Ce^{kt}}.$$

Therefore,

$$0.3(1 + Ce^{kt}) = Ce^{kt_0}$$

so

$$0.3 = 0.7Ce^{kt_0}$$

and

$$e^{kt_0} = \frac{0.3}{0.7C}.$$

This shows that

$$t_0 = \frac{\ln\left(\frac{0.3}{0.7C}\right)}{k} = \frac{\ln\left(\frac{0.3}{0.7(0.0526315)}\right)}{0.07472} = 28.07$$

hours.

After 28.07 hours, 30% of the student population had heard the rumor.

◀

3.2 & 3.3 Volumes of Revolution

Volumes of Revolution: The Disk Method I

Let f be continuous on $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f, the x-axis and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi f(x)^2 dx.$$

Volumes of Revolution: The Shell Method

Let $a \geq 0$. Let f and g be continuous on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the y-axis is given by

$$V = \int_a^b 2\pi x(g(x) - f(x))\, dx.$$

Volumes of Revolution: The Disk Method II

Let f and g be continuous on $[a,b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi(g(x)^2 - f(x)^2)\, dx.$$

	Functions of x	Functions of y
Vertical line	Cylindrical Shells	washers/disks
Horizontal line	washers/disks	Cylindrical Shells

3.4 Arc length

Arc Length

Let f be continuously differentiable on $[a,b]$. Then the arc length S of the graph of f over the interval $[a,b]$ is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

4.1 - Introduction to Differential Equations

DEFINITION

Differential Equation

A *differential equation* is an equation involving an independent variable such as x, a function $y = y(x)$ and various derivatives of y. In general, we will write

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0.$$

A *solution* to the differential equation is a function φ such that

$$F(x, \varphi(x), \varphi'(x), \cdots, \varphi^{(n)}(x)) = 0.$$

The highest order of a derivative appearing in the equation is called the *order* of the differential equation.

NOTE

1) In this course, we will typically consider only first-order differential equations. Such DEs can be written in the form

$$y' = f(x, y).$$

A solution for a first-order differentiable equation is a function φ for which

$$\varphi'(x) = f(x, \varphi(x)).$$

2) The simplest first-order DE is the equation

$$y' = f(x).$$

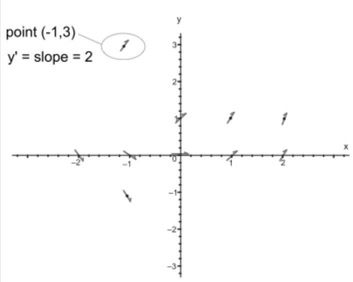
Hence $y = y(x)$ is a solution if and only if y is an antiderivative of f. Therefore, the solutions to this equation are given by

$$\int f(x)\, dx = F(x) + C$$

where F is any antiderivative of f and $C \in \mathbb{R}$ is an arbitrary constant. This shows that differential equations do not need to have unique solutions. In particular, each different choice of C results in a new solution. The constant C is called a *parameter* and the collection of solutions $\{F(x) + C \mid C \in \mathbb{R}\}$ is called a *one parameter family.* ◀

4.5.1 - Direction Fields

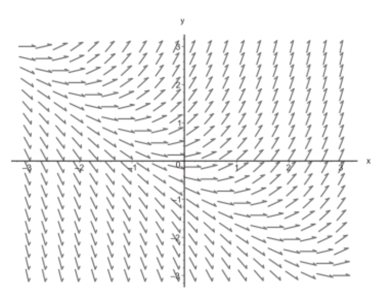
x	y	tangent line slope from DE $y' = x + y$
-2	0	$y' = -2 + 0 = -2$
-1	0	$y' = -1 + 0 = -1$
0	0	$y' = 0 + 0 = 0$
1	0	$y' = 1 + 0 = 1$
2	0	$y' = 2 + 0 = 2$
0	1	$y' = 0 + 1 = 1$
1	1	$y' = 1 + 1 = 2$
2	1	$y' = 2 + 1 = 3$
-1	3	$y' = -1 + 3 = 2$
-1	-1	$y' = -1 + -1 = -2$



The completed direction field for

$$y' = x + y$$

is shown.



Week 6: March 1 - 5

Textbook Sections/Topics:

- Section 5.1 - Introduction to Series
- Section 5.2 - Geometric Series
- Section 5.3 - Divergence Test
- Section 5.4 - Arithmetic of Series

5.1 - Introduction to Series

DEFINITION **Series** **Definition: Series**

Given a sequence $\{a_n\}$, the *formal* sum

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

is called a *series*. The series is called *formal* because we have not yet given it a meaning numerically.

The a_n's are called the *terms* of the series. For each term a_n, the *index* of the term is n.

We will denote the series by

$$\sum_{n=1}^{\infty} a_n.$$

DEFINITION **Convergence of a Series** **Definition: Convergence of a Series**

Given a series

$$\sum_{n=1}^{\infty} a_n$$

partial Sum

for each $k \in \mathbb{N}$, we define the *k-th partial sum* S_k by

$$S_k = \sum_{n=1}^{k} a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence $\{S_k\}$ of partial sums converges. In this case, if $L = \lim_{k\to\infty} S_k$, then we write

$$\sum_{n=1}^{\infty} a_n = L$$

and assign the sum this value. Otherwise, we say that the series $\sum_{n=1}^{\infty} a_n$ *diverges*.

5.2 - Geometric Series

DEFINITION **Geometric Series** **Definition: Geometric Series**

A *geometric series* is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

The number r is called the *ratio* of the series.

THEOREM 1 **Geometric Series Test** **Theorem: Geometric Series Test**

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise.

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

5.3 - Divergence Test

THEOREM 2 **Divergence Test** **Theorem: Divergence Test**

Assume that $\sum_{n=1}^{\infty} a_n$ converges. Then

$$\lim_{n\to\infty} a_n = 0.$$

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

The Divergence Test gets its name because it can identify certain series as being divergent, **but it cannot show that a series converges.**

Question: If $\lim_{n\to\infty} a_n = 0$, does this mean that $\sum_{n=1}^{\infty} a_n$ converges?

We will see that the answer to the question above is: **No**, the fact that $\lim_{n\to\infty} a_n = 0$, does not mean that $\sum_{n=1}^{\infty} a_n$ converges.

5.4 - Arithmetic of Series

THEOREM 3 **Arithmetic for Series I** **Theorem: Arithmetic for Series I**

Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

1. The series $\sum_{n=1}^{\infty} ca_n$ converges for every $c \in \mathbb{R}$ and
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$
2. The series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and
$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

THEOREM 4 **Arithmetic for Series II** **Theorem: Arithmetic for Series II**

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ also converges for **each** j.
2. If $\sum_{n=j}^{\infty} a_n$ converges for **some** j, then $\sum_{n=1}^{\infty} a_n$ converges.

In either of these two cases,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{j-1} + \sum_{n=j}^{\infty} a_n.$$

5.5 - Positive Series

DEFINITION **Monotonic Sequences**

Given a sequence $\{a_n\}$, we say that the sequence is

i) *non-decreasing* if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$.

ii) *increasing* if $a_{n+1} > a_n$ for every $n \in \mathbb{N}$.

iii) *non-increasing* if $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$.

iv) *decreasing* if $a_{n+1} < a_n$ for every $n \in \mathbb{N}$.

We say that $\{a_n\}$ is *monotonic* if it satisfies one of these four conditions.

DEFINITION **Positive Series**

We call a series $\sum\limits_{n=1}^{\infty} a_n$ *positive* if the terms $a_n \geq 0$ for all $n \in \mathbb{N}$.

THEOREM 5 **Monotone Convergence Theorem (MCT)**

Let $\{a_n\}$ be a non-decreasing sequence.

1. If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = lub(\{a_n\})$.
2. If $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞.

In particular, $\{a_n\}$ converges if and only if it is bounded above.

THEOREM 6 **Comparison Test for Series**

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$.

1. If $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges.

THEOREM 7 **Limit Comparison Test (LCT)**

Assume that $a_n > 0$ and $b_n > 0$ for each $n \in \mathbb{N}$. Assume also that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where either $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 < L < \infty$, then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} b_n$ converges.
2. If $L = 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges. Equivalently, if $\sum\limits_{n=1}^{\infty} a_n$ diverges, then so does $\sum\limits_{n=1}^{\infty} b_n$.
3. If $L = \infty$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} b_n$ converges. Equivalently, if $\sum\limits_{n=1}^{\infty} b_n$ diverges, then so does $\sum\limits_{n=1}^{\infty} a_n$.

5.6 - Integral Test

THEOREM 9 **p-Series Test**

The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

THEOREM 8 **Integral Test for Convergence**

Assume that

1. f is continuous on $[1, \infty)$,
2. $f(x) > 0$ on $[1, \infty)$,
3. f is decreasing on $[1, \infty)$, and
4. $a_k = f(k)$.

For each $n \in \mathbb{N}$, let $S_n = \sum\limits_{k=1}^{n} a_k$. Then

i) For all $n \in \mathbb{N}$,
$$\int_1^{n+1} f(x)\,dx \leq S_n \leq a_1 + \int_1^n f(x)\,dx.$$

ii) $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

iii) In the case that $\sum\limits_{k=1}^{\infty} a_k$ converges, then
$$\int_1^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x)\,dx$$
and
$$\int_{n+1}^{\infty} f(x)\,dx \leq S - S_n \leq \int_n^{\infty} f(x)\,dx,$$
where $S = \sum\limits_{k=1}^{\infty} a_k$. (Note that by (ii), $\int_n^{\infty} f(x)\,dx$ exists.)

5.7 - Alternating Series

DEFINITION **Alternating Series**

A series of the form
$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$
or of the form
$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$
is said to be *alternating* provided that $a_n > 0$ for all n.

THEOREM 10 **Alternating Series Test (AST)**

Assume that

1. $a_n > 0$ for all n.
2. $a_{n+1} \leq a_n$ for all n.
3. $\lim\limits_{n\to\infty} a_n = 0$.

Then the alternating series
$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$
converges.

If $S_k = \sum\limits_{n=1}^{k} (-1)^{n-1} a_n$, then S_k approximates the sum $S = \sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ with an error that is at most a_{k+1}. That is
$$| S_k - S | \leq a_{k+1}.$$

5.8 - Absolute Versus Conditional Convergence

DEFINITION **Absolute vs Conditional Convergence**

onvergence

A series $\sum\limits_{n=1}^{\infty} a_n$ is said to *converge absolutely* if

$$\sum_{n=1}^{\infty} | a_n |$$

converges.

A series $\sum\limits_{n=1}^{\infty} a_n$ is said to *converge conditionally* if

$$\sum_{n=1}^{\infty} | a_n |$$

diverges while

$$\sum_{n=1}^{\infty} a_n$$

converges.

DEFINITION **Rearrangement of a Series** Rearrangement of a Series

Given a series $\sum\limits_{n=1}^{\infty} a_n$ and a 1-1 and onto function $\phi : \mathbb{N} \to \mathbb{N}$, if we let

$$b_n = a_{\phi(n)},$$

then the series

$$\sum_{n=1}^{\infty} b_n$$

is called a *rearrangement* of $\sum\limits_{n=1}^{\infty} a_n$.

THEOREM 11 **Absolute Convergence Theorem** Absolute Convergence Theorem

If $\sum\limits_{n=1}^{\infty} | a_n |$ converges, then so does $\sum\limits_{n=1}^{\infty} a_n$.

Note: The sums $\sum\limits_{n=1}^{\infty} | a_n |$ and $\sum\limits_{n=1}^{\infty} a_n$ will converge to different values unless $a_n \geq 0$ for all n.

THEOREM 12 **Rearrangement Theorem** Rearrangement Theorem

1) Let $\sum\limits_{n=1}^{\infty} a_n$ be an absolutely convergent series. If $\sum\limits_{n=1}^{\infty} b_n$ is any rearrangement of $\sum\limits_{n=1}^{\infty} a_n$, then $\sum\limits_{n=1}^{\infty} b_n$ also converges and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

2) Let $\sum\limits_{n=1}^{\infty} a_n$ be a conditionally convergent series. Let $\alpha \in \mathbb{R}$ or $\alpha = \pm\infty$. Then there exists a rearrangement $\sum\limits_{n=1}^{\infty} b_n$ of $\sum\limits_{n=1}^{\infty} a_n$ such that

$$\sum_{n=1}^{\infty} b_n = \alpha.$$

Remark: In summary, whenever you must test a series with terms of mixed signs for convergence it is always a good idea to first check if the series converges absolutely.

5.9 - Ratio Test

THEOREM 14 **Polynomial vs Factorial Growth**

For any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Remark: This important limit tells us that exponentials are of a lower order of magnitude compared to factorials. That is, for any fixed $x_0 \in \mathbb{R}$, $| x_0 |^n \ll n!$

Note: We know that the series $\sum\limits_{n=0}^{\infty} r^n$ will diverge if $| r | = 1$. Therefore, since the conclusions of the Ratio Test are based on the Geometric Series Test, it might be surprising that if $L = 1$, the Ratio Test would not show that the series diverges. However, it is important to recognize that $\lim\limits_{n\to\infty} | \frac{a_{n+1}}{a_n} | = 1$ does not actually mean that $| a_{N+k} | = | a_N |$ for large N as would be the case if the ratio was exactly 1.

The next two examples show that when $L = 1$ we could have either convergence or divergence.

Fact: If $p(x) = a_0 + a_1 x + \cdots + a_k x^k$ and $q(x) = b_0 + b_1 x + \cdots + b_m x^m$ are two polynomials, then the Ratio Test will *always fail* for the series

$$\sum_{n=1}^{\infty} \frac{p(n)}{q(n)}$$

The following is a summary of what we have learned about the order of magnitude of various functions:

$$\ln(n) \ll n^p \ll x^n \ll n! \ll n^n$$

for $| x | > 1$.

Therefore,

$$\frac{1}{n^n} \ll \frac{1}{n!} \ll \frac{1}{x^n} \ll \frac{1}{n^p} \ll \frac{1}{\ln(n)}.$$

THEOREM 13 **Ratio Test**

Given a series $\sum\limits_{n=0}^{\infty} a_n$, assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum\limits_{n=0}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum\limits_{n=0}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Remarks:

1) If $0 \leq L < 1$, the Ratio Test shows that the given series converges absolutely and hence that the original series also converges.
2) If $\lim\limits_{n\to\infty} | \frac{a_{n+1}}{a_n} | = L$ exists with $L \neq 1$, then the series $\sum\limits_{n=0}^{\infty} a_n$ behaves like the geometric series $\sum\limits_{n=0}^{\infty} L^n$ as far as convergence is concerned.
3) While the Ratio Test is one of the most important tests for convergence, we will see that it cannot detect convergence or divergence for many of the series we have seen so far. In fact, it can only detect convergence if the terms a_n approach 0 very rapidly, and it can only detect divergence if $\lim\limits_{n\to\infty} |a_n| = \infty$. This means that the Ratio Test is appropriate for a very special class of series.

5.10 - Root test

THEOREM 15 **Root Test**

Given a series $\sum\limits_{n=1}^{\infty} a_n$, assume that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

6.1 - Power Series

DEFINITION

Power Series Power series

A power series centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where x is considered a variable and the value a_n is called the *coefficient* of the term $(x-a)^n$.

Interval and Radius of Convergence

DEFINITION

Interval and Radius of Convergence

Given a power series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$, the set

Ignore this !

$$I = \{x_0 \mid \sum_{n=0}^{\infty} a_n (x_0 - a)^n \text{ converges}\}$$

is an interval centered at $x = a$ which we call *the interval of convergence* for the power series.

Let

$$R := \begin{cases} lub(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded,} \\ \infty & \text{if } I \text{ is not bounded.} \end{cases}$$

Then R is called the *radius of convergence* of the power series.

R tell us how far we can deviate from **a** and still maintain convergence

THEOREM 1

Fundamental Convergence Theorem for Power Series

Fundamental Convergence Theorem for Power Series

Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ centered at $x = a$, let R be the radius of convergence.

1. If $R = 0$, then $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges for $x = a$ but it diverges for all other values of x.
2. If $0 < R < \infty$, then the series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges absolutely for every $x \in (a-R, a+R)$ and diverges if $| x - a |> R$.
3. If $R = \infty$, then the series $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges absolutely for every $x \in \mathbb{R}$.

In particular, $\sum_{n=0}^{\infty} a_n (x-a)^n$ converges on an interval that is centered at $x = a$ which may or may not include one or both of the endpoints.

Remark: If $0 < R < \infty$, then there are four possibilities for the interval of convergence I.

1) $I = (a-R, a+R)$ Example: $\sum_{n=0}^{\infty} x^n \Rightarrow I = (-1, 1)$.

2) $I = [a-R, a+R)$ Example: $\sum_{n=1}^{\infty} \frac{x^n}{n} \Rightarrow I = [-1, 1)$.

3) $I = (a-R, a+R]$ Example: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \Rightarrow I = (-1, 1]$.

4) $I = [a-R, a+R]$ Example: $\sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow I = [-1, 1]$.

Key Note: Once R is determined, you need to test the endpoints separately.

6.1.1 - Finding the Radius of Convergence

Test for the Radius of Convergences

THEOREM 2

Test for the Radius of Convergence

Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series for which

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $0 \leq L < \infty$ or $L = \infty$. Let R be the radius of convergence of the power series.

1. If $0 < L < \infty$, then $R = \frac{1}{L}$.
2. If $L = 0$, then $R = \infty$.
3. If $L = \infty$, then $R = 0$.

Equivalence of Radius of Convergence

THEOREM 3

Equivalence of Radius of Convergence

Let p and q be non-zero polynomials where $q(n) \neq 0$ for $n \geq k$. Then the following series have the same radius of convergence:

1. $\sum_{n=k}^{\infty} a_n (x-a)^n$
2. $\sum_{n=k}^{\infty} \frac{a_n p(n)(x-a)^n}{q(n)}$

However, they may have different intervals of convergence.

Key Observation: The series

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n(n^2+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

have the same radius of convergence!

6.2 - Functions Represented by Power Series

DEFINITION

Functions Represented by a Power Series

Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series with radius of convergence $R > 0$. Let I be the interval of convergence for $\sum_{n=0}^{\infty} a_n (x-a)^n$. Let f be the function defined on the interval I by the formula

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for each $x \in I$.

We say that the function $f(x)$ *is represented by the power series* $\sum_{n=0}^{\infty} a_n (x-a)^n$ *on* I.

Abel's Theorem, Continuity of Power series

THEOREM 4

Abel's Theorem: Continuity of Power Series

Assume that the power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ has interval of convergence I. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for each $x \in I$. Then $f(x)$ is continuous on I.

6.2.1 - Building Power Series Representations

Addition of power Series

THEOREM 5 **Addition of Power Series**

Assume that f and g are represented by power series centered at $x = a$ with

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n,$$

respectively.

Assume also that the radii of convergence of these series are R_f and R_g with intervals of convergence I_f and I_g. Then

$$(f+g)(x) = \sum_{n=0}^{\infty}(a_n + b_n)(x-a)^n.$$

Moreover, if $R_f \neq R_g$, then the radius of convergence of the power series representing $f+g$ is $R = \min\{R_f, R_g\}$ and the interval of convergence is $I = I_f \cap I_g$.

If $R_f = R_g$, then $R \geq R_f$.

THEOREM 6 **Multiplication of a Power Series by $(x-a)^m$**

Multiplication of a power Series

Assume that f is represented by a power series centered at $x = a$ as

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

with radius of convergence R_f and interval of convergence I_f.

Assume that $h(x) = (x-a)^m f(x)$ where $m \in \mathbb{N}$. Then $h(x)$ can also be represented by a power series centered at $x = a$ with

$$h(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n+m}$$

Moreover, the series that represents h has the same radius of convergence and the same interval of convergence as the series that represents f.

power Series of Composite Function

THEOREM 7 **Power Series of Composite Functions**

Assume that f has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with radius of convergence R_f and interval of convergence I_f. Let $h(x) = f(c \cdot x^m)$ where c is a non-zero constant. Then h has a power series representation centered at $x = 0$ of the form

$$h(x) = f(c \cdot x^m) = \sum_{n=0}^{\infty}(a_n \cdot c^n)x^{mn}$$

The interval of convergence is

$$I_h = \{x \in \mathbb{R} \mid c \cdot x^m \in I_f\}$$

and the radius of convergence is $R_h = \sqrt[m]{\frac{R_f}{|c|}}$ if $R_f < \infty$ and $R_h = \infty$ otherwise.

6.3 - Differentiation of Power Series

The Formal Derivative of a power Series

DEFINITION **The Formal Derivative of a Power Series**

Given a power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$, the *formal derivative* is the series

$$\sum_{n=0}^{\infty} na_n(x-a)^{n-1} = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}.$$

THEOREM 8 **Term-by-Term Differentiation of Power Series**

Assume that the power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and for each $x \in (a-R, a+R)$,

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}.$$

The interval of convergence may be different.

THEOREM 9 **Uniqueness of Power Series Representations**

Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$ where $R > 0$. Then

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

In particular, if

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n,$$

then

$$b_n = a_n$$

for each $n = 0, 1, 2, 3, \cdots$.

6.4 - Integration of Power Series

DEFINITION **Formal Antiderivative of a Power Series**

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, we define *the formal antiderivative* to be the power series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

where C is an arbitrary constant.

THEOREM 10 **Term-by-Term Integration of Power Series**

Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for every $x \in (a-R, a+R)$. Then the series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

also has radius of convergence R and if

$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

then $F'(x) = f(x)$.

Furthermore, if $[c, b] \subset (a-R, a+R)$, then

$$\begin{aligned} \int_c^b f(x)\,dx &= \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n \, dx \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot \left((b-a)^{n+1} - (c-a)^{n+1}\right) \end{aligned}$$

Important Note: It may seem perfectly natural that we are also able to integrate term-by-term functions that are represented by a power series. In general, if

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

for each $x \in [a, b]$, then we might hope that

$$\int_a^b F(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Fact: If we do not make any additional assumptions about the nature of the functions f_n or about how the series converges, then it is possible that the function F need not even be integrable on $[a, b]$ even if all of the f_n's are.

6.5 - Review of Taylor Polynomials

DEFINITION **Taylor Polynomials**

Assume that f is n-times differentiable at $x = a$. The *n-th degree Taylor polynomial for f centered at $x = a$* is the polynomial

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

NOTE

A remarkable property about $T_{n,a}$ is that for any k between 0 and n,

$$T_{n,a}^{(k)}(a) = f^{(k)}(a).$$

That is, $T_{n,a}$ encodes not only the value of $f(x)$ at $x = a$ but all of its first n derivatives as well. Moreover, this is the *only* polynomial of degree n or less that does so!

6.6 - Taylor's Theorem and Errors in Approximations

DEFINITION **Taylor Remainder**

Assume that f is n times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x).$$

$R_{n,a}(x)$ is called the *n-th degree Taylor remainder function centered at $x = a$.*

The error in using the Taylor polynomial to approximate f is given by

$$\textbf{Error} = | R_{n,a}(x) | \ .$$

THEOREM 12 **Taylor's Approximation Theorem I**

Assume that $f^{(k+1)}$ is continuous on $[-1, 1]$. Then there exists a constant $M > 0$ such that

$$| f(x) - T_{k,0}(x) | \leq M \, | x |^{k+1}$$

or equivalently that

$$-M \, | x |^{k+1} \leq f(x) - T_{k,0}(x) \leq M \, | x |^{k+1}$$

for each $x \in [-1, 1]$.

THEOREM 11 **Taylor's Theorem**

Assume that f is $n+1$-times differentiable on an interval I containing $x = a$. Let $x \in I$. Then there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

We will make **three important observations about Taylor's theorem.**

1) First, since $T_{1,a}(x) = L_a(x)$, when $n = 1$ the absolute value of the remainder $R_{1,a}(x)$ represents the error in using the linear approximation. Taylor's Theorem shows that for some c,

$$| R_{1,a}(x) | = | \frac{f''(c)}{2}(x-a)^2 |.$$

This shows explicitly how the error in linear approximation depends on the potential size of $f''(x)$ and on $| x - a |$, the distance from x to a.

2) The second observation involves the case when $n = 0$. In this case, the theorem requires that f be differentiable on I and its conclusion states that for any $x \in I$ there exists a point c between x and a such that

$$f(x) - T_{0,a}(x) = f'(c)(x-a).$$

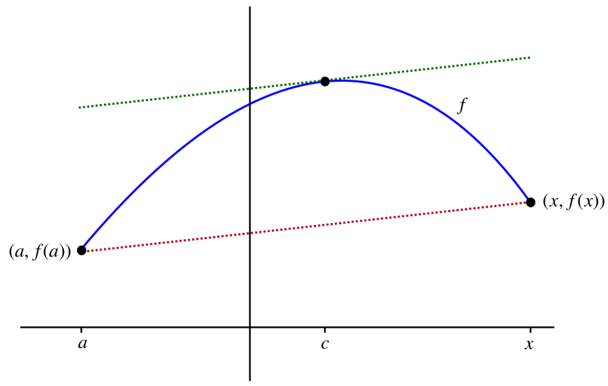
But $T_{0,a}(x) = f(a)$, so we have

$$f(x) - f(a) = f'(c)(x-a).$$

Dividing by $x - a$ shows that there is a point c between x and a such that

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

This is exactly the statement of the Mean Value Theorem. Therefore, Taylor's Theorem is really a higher-order version of the MVT.



3) Finally, Taylor's Theorem does not tell us how to find the point c, but rather that such a point exists. It turns out that for the theorem to be of any value, we really need to be able to say something intelligent about how large $| f^{(n+1)}(c) |$ might be without knowing c. For an arbitrary function, this might be a difficult task since higher order derivatives have a habit of being very complicated. However, the good news is that for some of the most important functions in mathematics, such as $\sin(x)$, $\cos(x)$, and e^x, we can determine roughly how large $| f^{(n+1)}(c) |$ might be and in so doing, show that the estimates obtained for these functions can be extremely accurate.

6.7 - Introduction to Taylor Series

DEFINITION **Taylor Series**

Assume that f has derivatives of all orders at $a \in \mathbb{R}$. The series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the *Taylor series* for f centered at $x = a$.

We write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In the special case where $a = 0$, the series is referred to as the *Maclaurin series* for f.

Remark:

Up until now, we have started with a function that was represented by a power series on its interval of convergence. In this case, the series that represents the function must be the Taylor Series.

However, suppose that f is any function for which $f^{(n)}(a)$ exists for each n. Then we can build the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

However, we do not know the following:

1) For which values of x does the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converge?

2) If the series converges at x_0, is it true that

$$f(x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x_0-a)^n \ ?$$

6.8 - convergence of Taylor series

Remark: Before we present the next example we need to recall the following limit which we previously established as a consequence of the Ratio Test.

Let $x_0 \in \mathbb{R}$. Then

$$\lim_{k\to\infty} \frac{M\,|\,x_0\,|^k}{k!} = 0.$$

◂

Remark: Notice that in each of the previous examples that if either $f(x) = \cos(x)$ or $f(x) = \sin(x)$, then the function f had the property that for any $k = 0, 1, 2, 3, \ldots$ and for each $x \in \mathbb{R}$, then

$$\left|f^{(k)}(x)\right| \le 1.$$

◂

THEOREM 13 **Convergence Theorem for Taylor Series**

Assume that $f(x)$ has derivatives of all orders on an interval I containing $x = a$. Assume also that there exists an M such that

$$|\,f^{(k)}(x)\,| \le M$$

for all k and for all $x \in I$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in I$.

6.9 - Binomial Series

Remark: Consider the expression

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Typically we are only concerned with the case where $k \in \{0, 1, 2, \ldots, n\}$. But the expression actually makes sense for any $k \in \mathbb{N} \cup \{0\}$. If $k > n$, then one of the terms in the expression

$$n(n-1)(n-2)\cdots(n-k+1)$$

will be 0 and so

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = 0.$$

Consequently,

$$\begin{aligned}(1+x)^n &= 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}x^k\end{aligned}$$

This leaves us to make the rather strange observation that the polynomial function $(1+x)^n$ is actually represented by the power series

$$1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}x^k$$

In other words, $1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}x^k$ is the Taylor Series centered at $x = 0$ for the function $(1+x)^n$.

By itself the observation above does not tell us anything new about the function $(1+x)^n$. However it does give us an important clue towards answering the following question.

THEOREM 14 **Binomial Theorem**

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

In particular, when $a = 1$ we have

$$(1+x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}x^k.$$

DEFINITION **Generalized Binomial Coefficients and Binomial Series**

Let $\alpha \in \mathbb{R}$ and let $k \in \{0, 1, 2, 3, \ldots\}$. Then we define the generalized binomial coefficient

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

if $k \neq 0$ and

$$\binom{\alpha}{0} = 1.$$

We also define the generalized binomial series for α to be the power series

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

THEOREM 15 **Generalized Binomial Theorem**

Let $\alpha \in \mathbb{R}$. Then for each $x \in (-1, 1)$ we have that

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

6.10 - Application of Taylor Series

Note: This series representation for $\arctan(x)$ is called the *Gregory's series* after the Scottish mathematician of the same name. The famous series expansion for π which we derived from Gregory's series is called *Leibniz's formula for π*. ◂